Geophysical Journal International

Geophys. J. Int. (2019) **219,** 1939–1947 doi: 10.1093/gji/ggz407
Advance Access publication 2019 September 11
GJI Seismology

The generalized standard-linear-solid model and the corresponding viscoacoustic wave equations revisited

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Accepted 2019 September 10. Received 2019 September 1; in original form 2019 March 3

SUMMARY

The generalized standard-linear-solid model, also called the Zener model, is widely used in viscoacoustic/viscoelastic wavefield forward and inverse modelling because the wave equations in this model can be written in differential equation form, which can be solved efficiently by time-domain numerical methods such as finite-difference method, spectral element method, etc. For this model, however, two different expressions for the relaxation function (or complex modulus) appear in the literature somewhat confusingly. In addition to this confusion, the time- and frequency-domain versions of the wave equations for the generalized standard-linear-solid model are scattered throughout the literature. Here, we revisit the generalized standard-linear-solid model and seek to overcome the confusion concerning the expression for the relaxation function (or modulus). We present a unified approach to derive the viscoacoustic wave equations. We start with the time- and frequency-domain formulations separately to derive two sets of viscoacoustic wave equations. All these viscoacoustic wave equations are expressed in a simple and compact form. The two sets of viscoacoustic wave equations are equivalent to each other. The proposed method to derive the appropriate viscoacoustic wave equations can be extended to derive wave equations for other dissipative media.

Key words: Elasticity and anelasticity; Acoustic properties; Computational seismology; Seismic attenuation; Wave propagation.

1 INTRODUCTION

Incorporation of attenuation and dispersion into the seismic wave equation is needed to properly describe wave propagation in the near-surface zone of the Earth. The viscoacoustic wave equation is widely employed for forward and inverse modelling problems in seismology. The constitutive equation for a viscoacoustic/viscoelastic medium is generally expressed by a Riemann–Stieltjes convolution integral in mathematics (Apostol 1974). However, such an integral limits the application of the viscoacoustic/viscoelastic wave equation, because calculating the convolution integral requires the complete time history of the wavefield, which increases significantly the computational cost.

The standard-linear-solid (SLS) model (also called the Zener model) is a classic mechanical representation of anelastic behaviour—it comprises a series combination of a spring and a Kelvin–Voigt solid (Fig. 1a) or a parallel combination of a spring and a Maxwell solid (Fig. 2a). The dispersion and attenuation are characterized by the phase velocity and dissipation factor (i.e. the inverse of quality factor), respectively. In a 1-D SLS model, the phase velocity increases with frequency and reaches its minimum and maximum values at zero and infinite frequencies, respectively, whereas the dissipation factor is bell-shaped with the maximum value at a specific frequency determined by the stress and strain relaxation times, and is zero at both zero and infinite frequency. The quantitative description of the phase velocity and dissipation factor can be found in Carcione (2014).

The generalized SLS model comprises multiple such SLS elements in parallel (Figs 1b and 2b) and is widely used to simulate the nearly constant Q behaviour of rocks (e.g. Emmerich & Korn 1987; Blanch *et al.* 1995; Blanc *et al.* 2016). The viscoacoustic/viscoelastic wave equation based on the generalized SLS model can be written in differential equation form, which can be effectively solved by many time-domain numerical methods such as the finite-difference method (e.g. Carcione *et al.* 1988b) and the spectral-element method (e.g. Komatitsch & Tromp 1999). A detailed introduction to time-domain numerical methods for wave equations can be found in Carcione *et al.* (2002) and Igel (2017). However, the geophysical literature shows two different expressions for the relaxation function (or modulus) in the generalized SLS model. The first expression can be found in, for example, Liu *et al.* (1976), Carcione *et al.* (1988a,b,c), Blanch *et al.* (1995) and Komatitsch & Tromp (1999). Compared with the first expression, the second expression has an extra factor $1/N$ outside the summation

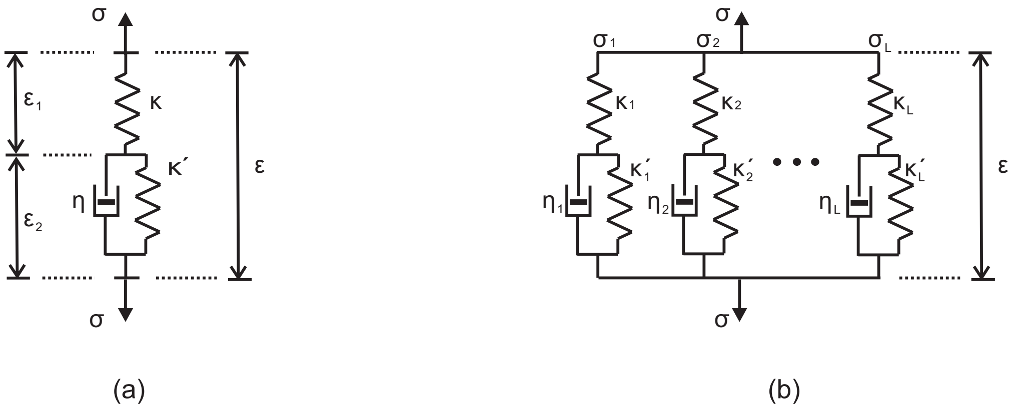

Figure 1. The Kelvin representation of the SLS model and its generalization. Diagram (a) shows the SLS model as a series combination of a Kelvin–Voigt solid and a spring. Diagram (b) shows the generalized SLS model as a parallel combination of multiple SLS elements. In diagram (a), ϵ and σ denote the total strain and stress, respectively, ϵ_1 and k denote the strain and modulus of the single spring, respectively, ϵ_2 and η denote the strain and viscosity of the dashpot, respectively, and $k^{'}$ denotes the modulus of the spring in the Kelvin–Voigt solid. The meaning of the symbols in diagram (b) is similar to that in diagram (a), and the subscripts denote the indexed SLS elements.

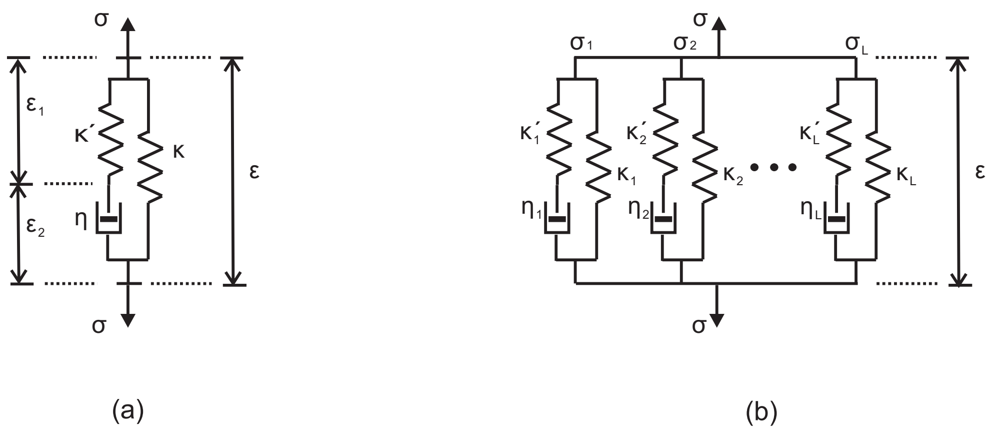

Figure 2. The Maxwell representation of the SLS model and its generalization. Diagram (a) shows the SLS model as a parallel combination of a Maxwell solid and a spring. Diagram (b) shows the generalized SLS model as a parallel combination of multiple SLS elements. In diagram (a), ϵ and σ denote the total strain and stress, respectively, ϵ_1 and $k^{'}$ denote the strain and modulus of the spring in the Maxwell solid, respectively, ϵ_2 and η denote the strain and viscosity of the dashpot in the Maxwell solid, respectively, and k denotes the modulus of the single spring. The meaning of the symbols in diagram (b) is similar to that in diagram (a), and the subscripts denote the indexed SLS elements.

term. The second expression can be found in, for example, Casula & Carcione (1992), Carcione (2014) and Bai & Tsvankin (2016). More confusingly, the parameter τ_{ϵ_l} in both expressions is assigned the same meaning in the references mentioned above.

The aim of this paper is to unify the expressions for the relaxation function and modulus for the generalized SLS model, and derive the corresponding viscoacoustic wave equations in differential-equation form from a unified perspective. For the generalized SLS model, we show two forms of the modulus and relaxation function, of which the relaxed modulus and relaxation times are related to the mechanical properties of the SLS elements. In this way, we link the existing two expressions for the generalized SLS relaxation function together and demonstrate their equivalence. For the generalized SLS model, we then derive two sets of viscoacoustic wave equations in differential-equation form. Each set includes three different viscoacoustic wave equations. We also demonstrate that the two sets of viscoacoustic wave equations can be derived from the time- and frequency-domains separately, and they are equivalent to each other.

The key to derive the viscoacoustic/viscoelastic wave equations in differential equation form for the generalized SLS model is to transform the constitutive equations into differential equation form. Carcione *et al.* (1988a,b) started with the time-domain constitutive equation to derive the viscoacoustic wave equation in differential equation form. Carcione *et al.* (1988c) used a similar method to derive the viscoelastic wave equation in differential equation form. By contrast, Emmerich & Korn (1987) started with the frequency-domain constitutive equation and derived the viscoacoustic wave equation in differential equation form. Dhemaied *et al.* (2011) worked in the frequency-domain but derived an alternative viscoelastic isotropic wave equation in differential equation form. A frequency-domain approach was chosen by Hao & He (2013) who showed two different forms of the viscoelastic orthorhombic wave equations for fractured media.

Since we need to switch freely between the time- and frequency-domain, we begin by defining the Fourier transform and its inverse. The Fourier transform of a temporal signal *f*(*t*) is written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt, \tag{1}$$

where t is time and ω is angular frequency.

The inverse Fourier transform of the frequency-domain signal $\hat{f}(\omega)$ is written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega t}\,d\omega. \tag{2}$$

It is noteworthy that the above definition of the Fourier transform and its inverse is consistent with the ones in Cerveny (2001, eq. A.1.2) and Hudson (1980, eq. 9.44), but different from the ones in Carcione *et al.* (1988a,b,c). For our definition, the first temporal derivative 'd/d*t*' corresponds to '$-i\omega$' in the frequency domain. As a result, the imaginary part of a complex stiffness coefficient in a viscoacoustic/viscoelastic medium is negative for a positive frequency, but it will be convenient to study the seismic rays in such a medium, because the imaginary part of complex traveltimes is always positive for positive frequencies (e.g. Cerveny & Psencik 2009; Hao & Alkhalifah 2017a,b).

In this paper, we assume the medium density to be constant so that we may obtain simple expressions for the viscoacoustic wave equations. For notational convenience, we do not show the spatial coordinates as arguments of the relaxation function or the bulk modulus, but our result is valid for the generalized SLS models with heterogeneous velocities and relaxation times.

2 THE GENERALIZED SLS MODEL

In this section, we show the Kelvin–Voigt and Maxwell representations of the SLS model and its generalization. We start with the configurations comprising springs and a dashpot to relate their mechanical properties to the parameters of the SLS model.

For convenience, we assume that the stress and strain are one-dimensional, bounded and causal. We also assume that the stress and strain are zero at time $t = 0$ and vary smoothly with time so that we do not need to consider the initial condition when taking into account the frequency-domain response. It is noteworthy that the two representations are not the only choices to describe the SLS model.

2.1 The SLS model

The time-domain constitutive equation for a one-dimensional viscoelastic model is written as (Gurtin & Sternberg 1962; Hudson 1980)

$$\begin{aligned}\sigma(t) &= \int_{-\infty}^{t} \psi(t-\tau)\,d\epsilon(\tau) \\ &= \psi(0+)\epsilon(t) + \int_{0}^{t} \dot{\psi}(t-\tau)\epsilon(\tau)\,d\tau,\end{aligned} \tag{3}$$

where σ and ϵ denote the stress and strain, respectively, both of which are bounded and causal. ψ denotes the relaxation function and the dot on $\dot{\psi}$ denotes the first temporal derivative. The term $\psi(0+)$ means the right-hand limit of the function ψ at zero time.

Performing the Fourier transform of the above equation, we derive the frequency-domain constitutive relation:

$$\hat{\sigma} = M\hat{\epsilon}, \tag{4}$$

where M denotes the complex modulus given by

$$M(\omega) = \psi(0+) + \int_{0}^{\infty} \dot{\psi}(t)e^{i\omega t}\,dt. \tag{5}$$

The relaxation function for the SLS model is (Carcione 2014)

$$\psi(t) = M_R\left[1 - \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right)e^{-\frac{t}{\tau_\sigma}}\right]H(t), \tag{6}$$

where *H*(*t*) denotes the Heaviside function.

Substituting eq. (6) into eq. (5) leads to the modulus for the SLS model:

$$M(\omega) = M_R\frac{1 - i\omega\tau_\epsilon}{1 - i\omega\tau_\sigma}, \tag{7}$$

where M_R denotes the relaxed modulus at zero frequency, τ_ϵ and τ_σ denote the strain and stress relaxation times, respectively. It is noteworthy that the minus sign in front of the imaginary unit i in the above equation corresponds to the definition of the Fourier transform (eq. 1).

In the following subsections, we utilize two classic mechanical representations to describe the SLS model and determine the relaxed modulus and the relaxation times.

2.1.1 The Kelvin–Voigt representation

As illustrated in Fig. 1(a), the Kelvin–Voigt representation comprises a series combination of a single spring and a Kelvin–Voigt solid. For the single spring, its modulus, stress and strain are denoted by k, σ_1 and ϵ_1, respectively. For the Kelvin–Voigt solid, the modulus and viscosity of the spring and dashpot are denoted by k' and η, respectively, and the spring and dashpot have the same strain denoted by ϵ_2. The stress acting on the single spring is equal to that acting on the Kelvin–Voigt solid. The total strain is equal to the sum of the strains of the single spring and the Kelvin–Voigt solid. From the above conditions, we obtain the following equations:

$$\begin{aligned}\sigma &= k\epsilon_1 = \eta\frac{d\epsilon_2}{dt} + k'\epsilon_2, \\ \epsilon &= \epsilon_1 + \epsilon_2.\end{aligned} \tag{8}$$

We transform eqs (8) into the frequency-domain and take into account the frequency-domain constitutive eq. (4) and the modulus expression (7). It follows that the relaxed modulus and the relaxation times are given by

$$\begin{aligned}M_R &= \frac{kk'}{k+k'}, \\ \tau_\epsilon &= \frac{\eta}{k'}, \\ \tau_\sigma &= \frac{\eta}{k+k'}.\end{aligned} \tag{9}$$

2.1.2 The Maxwell representation

As shown in Fig. 2(a), the Maxwell representation comprises a parallel combination of a single spring and a Maxwell solid. The modulus of the single spring is denoted by k and the modulus and viscosity of the spring and dashpot in the Maxwell solid are denoted by k' and η, respectively. The strain of the spring is the same as that of the Maxwell solid. In the Maxwell solid, the stress acting on the spring is equal to that acting on the dashpot. The total stress is equal to the sum of the stresses acting on the spring and the Maxwell solid. The total strain is equal to the sum of the strains of the spring and dashpot in the Maxwell solid. Describing mathematically the above conditions leads to the following equations:

$$\begin{aligned} \sigma &= k\epsilon + k'\epsilon_1, \\ k'\epsilon_1 &= \eta \frac{\mathrm{d}\epsilon_2}{\mathrm{d}t}, \\ \epsilon &= \epsilon_1 + \epsilon_2. \end{aligned} \tag{10}$$

Similar to the derivation of eqs (9), we derive from the above equations the relaxed modulus and relaxation times for the Maxwell representation,

$$\begin{aligned} M_R &= k, \\ \tau_\epsilon &= \eta \frac{k+k'}{kk'}, \\ \tau_\sigma &= \frac{\eta}{k'}. \end{aligned} \tag{11}$$

2.1.3 *The properties of the relaxed modulus and relaxation times*

From eqs (9) and (11), we make the following observations. For the Kelvin–Voigt and Maxwell representations of the SLS model, the relaxed modulus only depends on the moduli of the springs, whereas the dashpot only affects the stress and strain relaxation times; the strain relaxation time is always larger than the stress relaxation time, that is $\tau_\epsilon > \tau_\sigma$.

2.2 The generalized SLS model

Figs 1(b) and 2(b) show that the generalized SLS model can be obtained by a parallel combination of multiple SLS elements. The total stress is equal to the sum of the stresses acting on these SLS elements. Hence, the modulus for the generalized SLS model comprising L elements can be written as

$$M(\omega) = \sum_{l=1}^{L} M_{R_l} \frac{1 - \mathrm{i}\omega\tau_{\epsilon_l}}{1 - \mathrm{i}\omega\tau_{\sigma_l}}, \tag{12}$$

where index l denotes the lth mechanism, and the expressions for M_{R_l}, τ_{ϵ_l} and τ_{σ_l} are given by eq. (9) for the Kelvin–Voigt representation and by eq. (11) for the Maxwell representation.

2.2.1 *The first form of the relaxed modulus and relaxation function*

We rewrite the modulus (12) in the first form:

$$M(\omega) = M_R \left(1 - L + \sum_{l=1}^{L} \frac{1 - \mathrm{i}\omega\tilde{\tau}_{\epsilon_l}}{1 - \mathrm{i}\omega\tau_{\sigma_l}} \right), \tag{13}$$

with

$$M_R = \sum_{l=1}^{L} M_{R_l} \tag{14}$$

and

$$\tilde{\tau}_{\epsilon_l} = \frac{M_{R_l}}{M_R}\tau_{\epsilon_l} + \left(1 - \frac{M_{R_l}}{M_R}\right)\tau_{\sigma_l}. \tag{15}$$

As shown in eq. (15), the effective strain relaxation time $\tilde{\tau}_{\epsilon_l}$ describes the weighted average of the lth strain and stress relaxation times, where the weight M_{R_l}/M_R is always less than one for $L > 1$. Because for an SLS element $\tau_{\epsilon_l} > \tau_{\sigma_l}$ (see Section 2.1.3), we obtain $\tilde{\tau}_{\epsilon_l} > \tau_{\sigma_l}$.

For the SLS model, we know the corresponding relation between $\psi(t)$ (eq. 6) and $M(\omega)$ (eq. 7). For the generalized SLS model comprising multiple SLS elements, its relaxation function is a superposition of the relaxation functions of all these SLS elements. By analogy with the corresponding relation for the SLS model, for the generalized SLS model, the relaxation function corresponding to the relaxed modulus (eq. 12) is written as

$$\psi(t) = \sum_{l=1}^{L} M_{R_l} \left[1 - \left(1 - \frac{\tau_{\epsilon_l}}{\tau_{\sigma_l}}\right) \mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}} \right] H(t). \tag{16}$$

From eq. (15), we can obtain the expression for τ_{ϵ_l} in terms of $\tilde{\tau}_{\epsilon_l}$. Furthermore, we substitute this expression into eq. (16) and take into account eq. (14). After some algebraic manipulations, we finally obtain the first form of the relaxation function:

$$\psi(t) = M_R \left[1 - \sum_{l=1}^{L} \left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right) \mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}} \right] H(t). \tag{17}$$

2.2.2 *The second form of the relaxed modulus and relaxation function*

The relaxed modulus (12) is rewritten in the second form as:

$$M(\omega) = \frac{M_R}{L} \sum_{l=1}^{L} \frac{1 - \mathrm{i}\omega\tilde{\tau}'_{\epsilon_l}}{1 - \mathrm{i}\omega\tau_{\sigma_l}}, \tag{18}$$

with

$$\tilde{\tau}'_{\epsilon_l} = \tau_{\sigma_l} + \frac{M_{R_l}}{\overline{M}_R}(\tau_{\epsilon_l} - \tau_{\sigma_l}). \tag{19}$$

where $\overline{M}_R$ denotes the average relaxed modulus given by

$$\overline{M}_R = \frac{1}{L} \sum_{l=1}^{L} M_{R_l}. \tag{20}$$

Eq. (19) indicates that $\tilde{\tau}'_{\epsilon_l}$ describes the summation of the lth stress relaxation time and the weighted difference between the lth strain and stress relaxation times, where the weight is the ratio of the lth relaxed modulus to the average relaxed modulus. Because for an SLS element $\tau_{\epsilon_l} > \tau_{\sigma_l}$ (see Section 2.1.3), we obtain $\tilde{\tau}'_{\epsilon_l} > \tau_{\sigma_l}$ from eq. (19). Note that it is not appropriate to write $\tilde{\tau}'_{\epsilon_l}$ (eq. 19) in a similar form as eq. (15), because in this case the weight $M_{R_l}/\overline{M}_R$ is possibly larger than 1.

Referring to eqs (6), (7) and (18), we derive the second form of the relaxation function as:

$$\psi(t) = M_R \left[1 - \frac{1}{L} \sum_{l=1}^{L} \left(1 - \frac{\tilde{\tau}'_{\epsilon_l}}{\tau_{\sigma_l}}\right) \mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}} \right] H(t). \tag{21}$$

In the special case that the relaxed moduli for all the SLS elements are the same, that is, $M_{R_l} = \overline{M}_R$, eq. (19) then reduces to

$$\tilde{\tau}'_{\epsilon_l} = \tau_{\epsilon_l}. \tag{22}$$

In this special case, the relaxation function (eqs 21 with 22) is the same as eq. (2.198) of Carcione (2014).

2.2.3 The equivalence between the two forms

As we have shown, both modulus expressions (13) and (18) are obtained from eq. (12) without extra assumptions. Hence, the modulus (13) must be equivalent to the modulus (18). We only need to demonstrate that these two forms of the relaxation function are equivalent to each other. From eqs (15), (19) and (20), we relate the lth strain relaxation time in the first form to that in the second form,

$$\tilde{\tau}'_{\epsilon_l} = L\tilde{\tau}_{\epsilon_l} - (L-1)\tau_{\sigma_l}. \tag{23}$$

Substitution of eq. (23) into the second form of the relaxation function (eq. 21) leads to the first form (eq. 17). Conversely, we may derive the first form of the relaxation function from the second form. From eq. (23), we obtain the expression for $\tilde{\tau}_{\epsilon_l}$ in terms of $\tilde{\tau}'_{\epsilon_l}$. Furthermore, we substitute $\tilde{\tau}_{\epsilon_l}$ into the first form of the relaxation function (eq. 17). Finally, we may obtain the second form of the relaxation function (eq. 21).

3 BASIC EQUATIONS FOR VISCOACOUSTIC WAVES

The equation of motion for an elastic medium (ignoring the source term) is written as

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{24}$$

where σ_{ij} denote the components of the stress tensor, ρ denotes the density, u_i denote the components of the particle displacement and x_i denote the three Cartesian coordinates, with $i, j = 1, 2, 3$.

For an acoustic medium, the stress tensor is written as

$$\sigma_{ij} = -P\delta_{ij}, \tag{25}$$

where P is the pressure and δ_{ij} denotes the Kronecker delta.

Substituting eq. (25) into eq. (24), we obtain the acoustic equation of motion:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial P}{\partial x_i}. \tag{26}$$

By analogy with eq. (3), the time-domain constitutive equation for a general viscoacoustic medium is written as

$$\begin{aligned} -P(\mathbf{x}, t) &= \int_{-\infty}^{t} \psi(t-\tau)\mathrm{d}_\tau\theta(\mathbf{x}, \tau) \\ &= \psi(0+)\theta(\mathbf{x}, t) + \int_0^t \dot{\psi}(t-\tau)\theta(\mathbf{x}, \tau)\mathrm{d}\tau. \end{aligned} \tag{27}$$

where ψ and θ denote the bulk relaxation function and the cubical dilatation, respectively, both of which are assumed to be bounded and causal. The dot on $\dot{\psi}$ denotes the first temporal derivative and the term '$d_\tau\theta(\mathbf{x}, \tau)$' denotes the differential of the cubical dilatation with respect to time.

The cubical dilatation θ is equal to the divergence of the particle displacement, or the sum of the normal strains:

$$\theta = \nabla\cdot\mathbf{u}, \tag{28}$$

where ∇ denotes the gradient operator and $\mathbf{u}$ denotes the particle displacement, with components u_i.

By analogy with eq. (4), the frequency-domain viscoacoustic constitutive equation is given by

$$-\hat{P} = M\hat{\theta}, \tag{29}$$

where the hat on $\hat{P}$ and $\hat{\theta}$ denotes the Fourier transform and M denotes the bulk modulus given in eq. (5).

Referring to eqs (17) and (13), the bulk relaxation function and modulus for the generalized SLS model are summarized as follows:

$$\psi(t) = M_R\left[1 - \sum_{l=1}^{L}\left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right)\mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}}\right]H(t), \tag{30}$$

$$M(\omega) = M_R\left(1 - L + \sum_{l=1}^{L}\frac{1 - \mathrm{i}\omega\tilde{\tau}_{\epsilon_l}}{1 - \mathrm{i}\omega\tau_{\sigma_l}}\right), \tag{31}$$

where M_R denotes the relaxed bulk modulus. L denotes the number of relaxation mechanisms. τ_{σ_l} denotes the lth stress relaxation time and $\tilde{\tau}_{\epsilon_l}$ denotes the weighted average of the lth strain and stress relaxation times (eq. 15). $H(t)$ denotes the Heaviside function.

Alternative forms of the relaxation function and the modulus are given in eqs (21) and (18). These two forms of the relaxation function and the modulus were demonstrated to be equivalent to each other in the previous section.

4 THE FIRST SET OF VISCOACOUSTIC WAVE EQUATIONS

4.1 *Time-domain derivation*

Utilizing the commutativity of the convolution operation, an alternative form of the time-domain constitutive equation (eq. 27) is given by

$$-P(\mathbf{x}, t) = \psi(0+)\theta(\mathbf{x}, t) + \int_0^t \dot{\psi}(\tau)\theta(\mathbf{x}, t-\tau)\mathrm{d}\tau. \tag{32}$$

We let ψ_l denote the time-dependent term corresponding to the lth relaxation mechanism in the relaxation function for the generalized SLS model (eq. 30), that is,

$$\psi_l(t) = -M_R\left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right)\mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}}H(t), \tag{33}$$

such that the relaxation function is rewritten as

$$\psi(t) = M_R H(t) + \sum_{l=1}^{L}\psi_l(t). \tag{34}$$

Taking account of the above reformulation of the relaxation function, the first temporal derivative of the time-domain constitutive eq. (32) for the SLS model is written as

$$-\frac{\partial P}{\partial t} = \psi(0+)\frac{\partial\theta}{\partial t} + \sum_{l=1}^{L}\int_0^t \dot{\psi}_l(\tau)\dot{\theta}(\mathbf{x}, t-\tau)\mathrm{d}\tau, \tag{35}$$

with

$$\psi(0+) = M_R\left(1 - L + \sum_{l=1}^{L}\frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right), \tag{36}$$

$$\dot{\psi}_l(t) = \frac{M_R}{\tau_{\sigma_l}}\left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right)e^{-\frac{t}{\tau_{\sigma_l}}}H(t), \tag{37}$$

where we have used eqs (30) and (33) and assumed the initial cubical dilatation at $t = 0$ to be zero, that is, $\theta(\mathbf{x}, 0) = 0$.

Let r_l be an auxiliary variable to represent the integral in eq. (35), that is,

$$r_l = \int_0^t \dot{\psi}_l(\tau)\dot{\theta}(\mathbf{x}, t-\tau)\mathrm{d}\tau. \tag{38}$$

Hence, eq. (35) can be rewritten as

$$-\frac{\partial P}{\partial t} = \psi(0+)\frac{\partial \theta}{\partial t} + \sum_{l=1}^{L} r_l. \tag{39}$$

We use the commutativity of the convolution operation to rewrite eq. (38) as

$$r_l = \int_0^t \dot{\psi}_l(t-\tau)\dot{\theta}(\mathbf{x}, \tau)\mathrm{d}\tau. \tag{40}$$

Hence, the first temporal derivative of r_l is given by

$$\frac{\partial r_l}{\partial t} = \dot{\psi}_l(0)\frac{\partial \theta}{\partial t} - \frac{1}{\tau_{\sigma_l}} r_l, \tag{41}$$

where we have used the property $\partial\psi_l/\partial t = -\psi_l/\tau_{\sigma_l}$.

Eqs (39) and (41) form the time-domain constitutive equation for the generalized SLS model in differential equation form. We substitute eq. (28) into these two equations and introduce the momentum density $\mathbf{J} = \rho\partial\mathbf{u}/\partial t$ to incorporate the density, which allows us to rewrite them in a different way. We may also rewrite the equation of motion (eq. 26) using the momentum density. Finally, all these operations lead to the first viscoacoustic wave equations for the generalized SLS model in differential equation form. From these viscoacoustic wave equations, we derive the second and third viscoacoustic wave equations. These viscoacoustic wave equations are summarized as follows.

The first viscoacoustic wave equations are given by

$$\begin{aligned} \frac{\partial P}{\partial t} &= -v_U^2 \nabla\cdot\mathbf{J} - \sum_{l=1}^{L} r_l, \\ \frac{\partial r_l}{\partial t} &= -s_l \nabla\cdot\mathbf{J} - \frac{1}{\tau_{\sigma_l}} r_l, \quad 1 \le l \le L, \\ \frac{\partial \mathbf{J}}{\partial t} &= -\nabla P, \end{aligned} \tag{42}$$

where v_U denotes the acoustic velocity in the unrelaxed (infinite frequency) state,

$$v_U = v_R \sqrt{1 - L + \sum_{l=1}^{L} \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}}, \tag{43}$$

and s_l denotes the density-normalized '$-\partial\psi_l/\partial t$' at $t = 0$,

$$s_l = \frac{v_R^2}{\tau_{\sigma_l}}\left(\frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}} - 1\right). \tag{44}$$

Here $v_R = \sqrt{M_R/\rho}$ denotes the acoustic velocity in the relaxed (zero frequency) state and s_l is always positive as explained after eq. (15).

The second viscoacoustic wave equations are given by

$$\begin{aligned} \frac{\partial P}{\partial t} &= -v_U^2 E - \sum_{l=1}^{L} r_l, \\ \frac{\partial r_l}{\partial t} &= -s_l E - \frac{1}{\tau_{\sigma_l}} r_l, \quad 1 \le l \le L, \\ \frac{\partial E}{\partial t} &= -\nabla^2 P, \end{aligned} \tag{45}$$

where $E = \nabla\cdot\mathbf{J}$ denotes the divergence of the momentum density and ∇^2 denotes the Laplacian operator.

The third, more compact, viscoacoustic wave equations are given by

$$\begin{aligned} \frac{\partial^2 P}{\partial t^2} &= v_U^2 \nabla^2 P - \sum_{l=1}^{L} w_l, \\ \frac{\partial w_l}{\partial t} &= s_l \nabla^2 P - \frac{1}{\tau_{\sigma_l}} w_l, \quad 1 \le l \le L, \end{aligned} \tag{46}$$

where $w_l = \partial r_l/\partial t$.

4.2 *Frequency-domain derivation*

We rewrite the bulk modulus for the generalized SLS model (eq. 31) as

$$M = M_0 + \sum_{l=1}^{L} \frac{M_l}{1 - i\omega\tau_{\sigma_l}}, \tag{47}$$

with

$$M_0 = M_R\left(1 - L + \sum_{l=1}^{L} \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right), \tag{48}$$

$$M_l = M_R\left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right). \tag{49}$$

If we let

$$\hat{r}'_l = -\frac{\mathrm{i}\omega M_l}{1 - \mathrm{i}\omega\tau_{\sigma_l}}\hat{\theta}, \tag{50}$$

then the frequency-domain constitutive eq. (29) can be rewritten as

$$-\hat{P} = M_0\hat{\theta} + \sum_{l=1}^{L} \frac{\hat{r}'_l}{(-i\omega)}. \tag{51}$$

Multiplying the above equation by '$-i\omega$', substituting the Fourier transform of the cubical dilatation (eq. 28) and taking the inverse Fourier transform (eq. 2), we may derive the following differential equation,

$$\frac{\partial P}{\partial t} = -M_0\frac{\partial \theta}{\partial t} - \sum_{l=1}^{L} r'_l, \tag{52}$$

where we recall that the frequency-domain factor '$-i\omega$' corresponds to the first temporal derivative $\partial/\partial t$ in the time-domain.

Moving the denominator of the fraction on the right-hand side of eq. (50) to the left side and performing the inverse Fourier transform, we may obtain another differential equation:

$$r'_l + \tau_{\sigma_l}\frac{\partial r'_l}{\partial t} = M_l\frac{\partial \theta}{\partial t}. \tag{53}$$

Furthermore, we substitute eqs (28), (48) and (49) with the relation $M_R = \rho v_R^2$ into the above two equations and take into account the definition of the momentum density for simplification. The remaining part is to rewrite the equation of motion (eq. 26) in terms of the momentum density, which is similar to the one shown in the previous subsection. Finally, we end up with the first viscoacoustic wave equation (42). The second and third viscoacoustic wave equations are not repeated either, since as a consequence of the first viscoacoustic wave equation, they are shown in eqs (45) and (46).

5 THE SECOND SET OF VISCOACOUSTIC WAVE EQUATIONS

5.1 *Time-domain derivation*

We let

$$P = \sum_{l=0}^{L} P_l, \tag{54}$$

such that the time-domain constitutive equation for the generalized SLS model (eqs 27 and 30) can be rewritten as

$$-P_0 = M_R\theta, \tag{55}$$

$$P_l = \psi_l'(0+)\theta + \int_0^t \dot{\psi}_l'(t-\tau)\theta(\mathbf{x},\tau)\mathrm{d}\tau, \quad 1 \le l \le L, \tag{56}$$

with

$$\psi_l'(t) = M_R\left(1 - \frac{\tilde{\tau}_{\epsilon_l}}{\tau_{\sigma_l}}\right)\mathrm{e}^{-\frac{t}{\tau_{\sigma_l}}} H(t). \tag{57}$$

Taking the derivative of eqs (55) and (56) with respect to t, we obtain

$$\frac{\partial P_0}{\partial t} = -M_R\frac{\partial\theta}{\partial t}, \tag{58}$$

$$\frac{\partial P_l}{\partial t} = \psi_l'(0+)\frac{\partial\theta}{\partial t} - \frac{1}{\tau_{\sigma_l}}P_l, \tag{59}$$

where we have used the property $\partial\psi_l'/\partial t = -\psi_l'/\tau_{\sigma_l}$.

We next adopt a similar approach to derive the second set of viscoacoustic wave equations. We express the relaxed modulus in eq. (58) as $M_R = \rho v_R^2$. We substitute the cubical dilatation (eq. 28) into eqs (58) and (59), take into account the momentum density $\mathbf{J} = \rho\partial\mathbf{u}/\partial t$ and assume the medium density to be constant. All these operations allow us to rewrite eqs (58) and (59) to involve the momentum density. The equation of motion (eq. 26) can be rewritten in a similar way. Such modifications to these equations lead to a viscoacoustic wave equation. We summarize the second set of viscoacoustic wave equations as follows.

The first viscoacoustic wave equations are

$$\begin{aligned}
P &= \sum_{l=0}^{L} P_l,\\
\frac{\partial P_0}{\partial t} &= -v_R^2\nabla\cdot\mathbf{J},\\
\frac{\partial P_l}{\partial t} &= -\tau_{\sigma_l}s_l\nabla\cdot\mathbf{J} - \frac{1}{\tau_{\sigma_l}}P_l, \quad 1 \le l \le L,\\
\frac{\partial \mathbf{J}}{\partial t} &= -\nabla P,
\end{aligned} \tag{60}$$

where s_l is given in eq. (44).

The second viscoacoustic wave equations are

$$\begin{aligned}
P &= \sum_{l=0}^{L} P_l,\\
\frac{\partial P_0}{\partial t} &= -v_R^2 E,\\
\frac{\partial P_l}{\partial t} &= -\tau_{\sigma_l}s_l E - \frac{1}{\tau_{\sigma_l}}P_l, \quad 1 \le l \le L,\\
\frac{\partial E}{\partial t} &= -\nabla^2 P.
\end{aligned} \tag{61}$$

The third viscoacoustic wave equations are

$$\begin{aligned}
P &= \sum_{l=0}^{L} P_l,\\
\frac{\partial^2 P_0}{\partial t^2} &= v_R^2\nabla^2 P,\\
\frac{\partial^2 P_l}{\partial t^2} &= \tau_{\sigma_l}s_l\nabla^2 P - \frac{1}{\tau_{\sigma_l}}\frac{\partial P_l}{\partial t}, \quad 1 \le l \le L.
\end{aligned} \tag{62}$$

5.2 *Frequency-domain derivation*

We rewrite the bulk modulus (eq. 31) as

$$M = M_R + \sum_{l=1}^{L}\frac{\mathrm{i}\omega M_l'}{1 - \mathrm{i}\omega\tau_{\sigma_l}}, \tag{63}$$

with

$$M_l' = M_R\left(\tau_{\sigma_l} - \tilde{\tau}_{\epsilon_l}\right). \tag{64}$$

Utilizing the frequency-domain constitutive equation (eq. 29), we let

$$-\hat{P}_0' = M_R\hat{\theta}, \tag{65}$$

$$-\hat{P}_l' = \frac{\mathrm{i}\omega M_l'}{1 - \mathrm{i}\omega\tau_{\sigma_l}}\hat{\theta}, \tag{66}$$

such that the pressure is split into various parts

$$\hat{P} = \sum_{l=0}^{L}\hat{P}_l'. \tag{67}$$

Transforming eq. (65) to the time domain, substituting eq. (28) into the result and taking the first temporal derivative, we obtain the following differential equation,

$$\frac{\partial P_0'}{\partial t} = -M_R\nabla\cdot\mathbf{v}, \tag{68}$$

where $\mathbf{v}$ denotes the particle velocity vector.

Moving the denominator of the fraction on the right-hand side of eq. (66) to the left-hand side, performing the inverse Fourier transform, and substituting eq. (28), we may obtain another differential equation,

$$P_l' + \tau_{\sigma_l}\frac{\partial P_l'}{\partial t} = M_l'\nabla\cdot\mathbf{v}, \quad 1 \le l \le L. \tag{69}$$

Taking into account the relation $M_R = \rho v_R^2$ and eqs (44) and (64), we may rewrite the above two equations and the equation of motion (eq. 26) to involve the momentum density. A combination of all these equations gives rise to the wave equation (60). As illustrated in the previous subsection, the other two forms of the wave equations can be obtained further as a consequence.

6 THE EQUIVALENCE BETWEEN THE TWO SETS OF VISCOACOUSTIC WAVE EQUATIONS

Comparing the first set of viscoacoustic wave equations (eqs 42, 45 and 46) with the second set of viscoacoustic wave equations (eqs 60–62), we find that the two sets of equations are equivalent to each other through the following relations:

$$r_l = \frac{1}{\tau_{\sigma_l}} P_l, \tag{70}$$

$$w_l = \frac{1}{\tau_{\sigma_l}} \frac{\partial P_l}{\partial t}. \tag{71}$$

We take the first temporal derivative of the first of the viscoacoustic wave equation (60), substitute the second and third of these viscoacoustic wave equations into the result, and subsequently make use of eqs (43), (44) and (70). Finally, we obtain the viscoacoustic wave equation (42). Similarly, we can transform the viscoacoustic wave equations (61) and (62) to the viscoacoustic wave equations (45) and (46), respectively. Conversely, we may substitute eqs (70) and (71) into the first set of viscoacoustic wave equations and split the pressure P into the summation form [referring to the first of the viscoacoustic wave equation (60), for instance]. After some algebraic manipulation, we may finally obtain the second set of viscoacoustic wave equations.

A source term is needed to implement our proposed viscoacoustic wave equations for seismic modelling. We may maintain the one-to-one equivalence between the two sets of viscoacoustic wave equations in the following way: we add the source term to the right side of the equations including $\partial P/\partial t$ and $\partial P_0/\partial t$ in the first and second sets of viscoacoustic wave equations, respectively.

7 DISCUSSION

Carcione (2014, p.94) mentioned that 'As stated before, some processes, as for example, grain-boundary relaxation, have a dissipation factor that is much broader than a single relaxation curve. It seems natural to try to explain this broadening with a distribution of relaxation mechanisms. This approach was introduced by Liu *et al.* (1976) to obtain a nearly constant quality factor over the seismic frequency range of interest. Strictly, their model cannot be represented by mechanical elements, since it requires a spring of negative constant (Casula & Carcione, 1992)'. Further, Carcione (2014, p.96) stated that 'The relaxation function obtained by Liu *et al.* (1976) lacks the factor $1/L$'. Here, the relaxation function obtained by Liu *et al.* (1976) is the same as the first form of the relaxation function we obtained (eq. 17). As demonstrated in Section 2.2.3, however, the first form of the relaxation function can be described by a parallel combination of multiple SLS elements in the Kelvin–Voigt/Maxwell representations (Figs 1b and 2b), and is equivalent to the second form, which is a general case of eq. (2.198) in Carcione (2014), because the strain relaxation time τ_{ϵ_l} in his eq. (2.198) is obtained by assuming the relaxed moduli of all SLS elements are equal to each other. Such an assumption is unnecessarily restrictive.

We use the frequency-domain formulations to express the relaxed modulus and relaxation times of the SLS model in terms of the mechanical properties of the springs and dashpot in the Kelvin–Voigt and Maxwell representations. Furthermore, the generalized SLS model is obtained by superposition of multiple SLS elements.

As an alternative, our derivation can also be carried out in the time-domain. We take the Kelvin–Voigt representation (Fig. 1a) as an example. From eqs (8), we may obtain a time-domain first-order differential equation in terms of σ and ϵ. Taking into account the initial condition that σ and ϵ are zero at $t = 0$, we may solve the differential equation for σ and rewrite its solutions in the form of eq. (27), from which we determine the relaxation function $\psi(t)$. Furthermore, we rewrite $\psi(t)$ in the form of eq. (6) to determine M_R, $\tilde{\tau}_\epsilon$ and τ_σ for the SLS model. The above approach can also be applied to the SLS model in the Maxwell representation. For the generalized SLS model composed of L parallel SLS elements (Figs 1b and 2b), its relaxation function is the sum of the relaxation functions of all SLS elements. Rewriting the relaxation function for the generalized SLS model in the form of eq. (30) gives rise to the expressions for M_R, $\tilde{\tau}_{\epsilon_l}$ and τ_{σ_l}.

Although this means that the time- and frequency-domain approaches are equivalent to each other, the frequency-domain approach is more convenient. The time-domain approach involves the exponential terms in the relaxation function, but the frequency-domain approach only requires some simple algebraic operations without the exponential terms.

All the wave equations derived in this paper correspond to the modulus given in eq. (13) and the relaxation function given in eq. (17). Alternative forms of the relaxation function and the modulus are given in eqs (21) and (18). Since in Section 2 we demonstrate that the two forms of the relaxation function and the modulus are equivalent to each other, we will not derive the wave equations from the second form.

8 CONCLUSIONS

The two forms of the relaxation function (or modulus) for the generalized SLS model are equivalent to each other. The viscoacoustic wave equations for the generalized SLS model are obtained from the unified approach by considering both the time- and frequency-domain formulations. The two sets of viscoacoustic wave equations are equivalent to each other. The analysis shown in this study can be extended to derive the viscoacoustic anisotropic wave equations, and the viscoelastic isotropic/anisotropic wave equations for the generalized SLS model.

ACKNOWLEDGEMENTS

The research was supported by the College of Petroleum Engineering and Geosciences at KFUPM. The authors are grateful to Professors Tariq Alkhalifah and Alexey Stovas for their constructive reviews.

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