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Analytic calculation of phase and group velocities of P-waves in orthorhombic media

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ABSTRACT

We have developed an approximate method to calculate the P-wave phase and group velocities for orthorhombic media. Two forms of analytic approximations for P-wave velocities in orthorhombic media were built by analogy with the five-parameter moveout approximation and the four-parameter velocity approximation for transversely isotropic media, respectively. They are called the generalized moveout approximation (GMA)-type approximation and the Fomel approximation, respectively. We have developed approximations for elastic and acoustic orthorhombic media. We have characterized the elastic orthorhombic media in Voigt notation, and we can describe the acoustic orthorhombic media by introducing the modified Alkhalifah's notation. Our numerical evaluations indicate that the GMA-type and Fomel approximations are accurate for elastic and acoustic orthorhombic media with strong anisotropy, and the GMA-type approximation is comparable with the approximation recently proposed by Sripanich and Fomel. Potential applications of the proposed approximations include forward modeling and migration based on the dispersion relation and the forward traveltime calculation for seismic tomography.

INTRODUCTION

An elastic orthorhombic medium is described by nine independent stiffness coefficients and the specification of three mutually orthogonal planes of mirror symmetry. In each symmetry plane, the medium exhibits transverse isotropy. In these kinds of media, the P-wave phase and group velocities are characterized by all nine independent density-normalized stiffness coefficients. Tsvankin's (1997) notation is widely used to parameterize elastic orthorhombic media (Appendix A). On the other hand, orthorhombic models under the acoustic assumption as an ideal case are practically useful for seismic modeling, imaging, and inversion based on P-wave traveltimes for orthorhombic media (Han and Xu, 2012; Xu and Zhou, 2014). For such models, the S-wave velocities along the three symmetry axes are assumed to be zero (Alkhalifah, 2003). Only six independent parameters are required to describe the P-wave phase and group velocities in these kinds of media. Alkhalifah's (2003) notation is normally used to parameterize the acoustic orthorhombic media (Appendix B). Besides, the weak anisotropy notation (Farra and Pšenčík, 2003) is also widely used in studying wave propagations in orthorhombic media.

For elastic and acoustic orthorhombic media, the P-wave phase velocity as an explicit function of the phase angles (including the polar and the azimuthal angles of phase propagation direction) can be exactly calculated by solving a cubic equation from the Christoffel equation with respect to the phase velocity squared (Schoenberg and Helbig, 1997). However, it is difficult to exactly calculate the magnitude of P-wave group velocity from the group angles (similar to the definition of phase angles, but corresponding to group propagation direction) in a general orthorhombic medium no matter whether the medium is elastic or acoustic. This is because there is no exact and explicit relation for finding the group velocity for a given ray-velocity direction. An approach to overcome this difficulty is to exactly calculate the magnitude and direction of the group velocity vector for each phase propagation direction by making use of ray-tracing equations (Červený [2001], pp. 149–151), then numerically map the variation of the group velocity versus the group direction.

Analytic representation for P-wave velocities is desirable for the purpose of practical applications in seismic exploration. For instance, from approximations for the phase velocity of P-waves, one can easily obtain the dispersion relation that is very important for the pseudowave modeling and reverse time migration (Song and Alkhalifah, 2013); analytic formulations for the group velocity of

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P-waves are useful for calculating the ray traveltime in seismic tomography (McMechan, 1983; Williamson, 1990; Eaton, 1993) and inverting for medium parameters in homogeneous media (Mahmoudian et al., 2014). Until now, several analytic approximations have been proposed for the phase and group velocities of P-waves in orthorhombic media: For general anisotropic media, linearizing methods and the first-order perturbation theory are widely used to obtain the analytic approximations for phase velocities in terms of stiffness coefficients (Backus, 1965; Jech and Pšenčík, 1989; Mensch and Rasolofosaon, 1997; Pšenčík and Gajewski, 1998). These analytic approximations require fewer parameters for orthorhombic media. For instance, for orthorhombic media, the phase velocity formula proposed by Pšenčík and Gajewski (1998) depends on six weak-anisotropy parameters. In their paper, they also present approximate formulas for NMO velocities of P-waves in general weakly anisotropic media. Farra (2001) extends the first-order perturbation technique (Jech and Pšenčík, 1989; Pšenčík and Gajewski, 1998; Mensch and Farra, 1999) to higher orders for phase velocities of elastic waves in a general anisotropic medium. Pšenčík and Vavrycuk (2002) and Farra (2004) propose the first-order formulas for group velocities of body waves in general anisotropic media. Farra and Pšenčík (2013) provide the first- and second-order approximations for group velocities and moveouts of body waves in general anisotropic media, although their formulas for moveout are numerically tested only for transversely isotropic media with a vertical symmetry axis (VTI).

For orthorhombic media, the first-order approximations for phase and group velocities of elastic waves were proposed by Song and Every (2000) and Song et al. (2001) for weakly orthorhombic media, respectively. Daley and Krebes (2004a, 2004b) start in a different way to derive the first-order approximations for the P-wave phase and group velocities in such media, which are identical to those proposed by Song and Every (2000) and Song et al. (2001). Pšenčík and Gajewski (1998) use the weak anisotropy notation to characterize orthorhombic media and derive the first-order approximation for the P-wave phase velocity. Mensch and Farra (1999) derive the first-order approximation for the P-wave eigenvalue from the Christoffel equation for orthorhombic media, which can be easily transformed to the analytic approximation for the P-wave phase velocity. As mentioned above, this first-order approximation was generalized by Farra (2001) to a higher order for elastic waves in general anisotropic media. Tsvankin (1997) derives the P-wave phase velocity approximation in terms of Thomsen-type parameters by assuming the weak anisotropy of orthorhombic media. The P-wave traveltime approximation for multilayered orthorhombic models is proposed by Xu et al. (2005) and Vasconcelos and Tsvankin (2006), and this approximation can be transformed to the analytic representation of the P-wave group velocity for orthorhombic media. Sripanich and Fomel (2014) recently propose an anelliptic approximation for P-wave phase and group velocities in orthorhombic media, and their formulation exhibits accurate results for orthorhombic media with strong anisotropy.

Because the three symmetry planes of an orthorhombic media exhibit transverse isotropy, all these approximations mentioned above can be simplified to calculate phase and group velocities of P-waves in transversely isotropic (TI) media. Apart from the approximations for orthorhombic media mentioned above, there are a few other approximations for phase and group velocities in TI media (Thomsen, 1986; Dellinger et al., 1993; Daley et al., 2004; Fomel, 2004; Berryman, 2008; Farra and Pšenčík, 2013). A recent paper by Sripanich and Fomel (2014) presents a new anelliptic approximation for the P-wave phase and group velocities in TI and orthorhombic media. As we said before, moveout formulas for a considered medium can be transformed to corresponding formulas for group velocity. Therefore, group velocity approximations can be found from the papers involving moveout approximations (Alkhalifah, 1998, 2000; Ursin and Stovas, 2006; Fomel and Stovas, 2010; Stovas, 2010). A recent review of nonhyperbolic moveout approximations for VTI media can be found in Golikov and Stovas (2012). The accuracy of approximations for TI media will not be numerically investigated here.

In this paper, we devise a method to analytically calculate phase and group velocities of P-waves in an orthorhombic medium. In this method, anelliptic functions depending on horizontal and vertical velocities are involved in building approximations for the phase and group velocity surfaces. Furthermore, we propose two types of approximations: One has the form of the generalized moveout approximation (GMA) (Fomel and Stovas, 2010; Stovas, 2010; Stovas and Fomel, 2012), and we call it the GMA-type approximation, and the other has the form of the Fomel (2004) approximation, and we call it the Fomel approximation. The GMA-type approximation is developed from the five-parameter approximation for the P-wave traveltime in the heterogeneous media, whereas the Fomel-type approximation arises from the four-parameter approximation for P-wave velocities in TI media.

To express phase and group velocities approximations for elastic and acoustic orthorhombic media, we adopt two different notations in this paper: The density-normalized stiffness coefficients c_{ij} in Voigt notation are involved in the phase and group velocities of P-waves for elastic orthorhombic media; the modified Alkhalifah notation (Appendix B) is used for acoustic orthorhombic media. Besides, Tsvankin's (1997) notation is used in the section "Numerical examples" for acoustic orthorhombic models. To describe the phase and group propagation directions of P-waves, it is assumed that the normals to the three symmetry planes of an orthorhombic medium coincide with the Cartesian coordinate system that obeys the right-hand rule, the $[x, y]$ plane is horizontal, and the z-axis always points downward.

The paper is organized as follows: First, we discuss the approximations for phase and group velocities of P-waves in the next two sections. Then, the comparison of accuracy between our approximations and other existing approximations is shown in numerical examples. Finally, we discuss the disadvantages of our approximations.

ANELLIPTIC APPROXIMATIONS FOR PHASE VELOCITY

The GMA-type approximation

By analogy with the formula of GMA for a horizontal VTI layer (Stovas, 2010), the GMA-type approximation for the P-wave phase velocity in orthorhombic media is defined as

$$v_{\mathrm{P}}^2(\theta,\varphi) = (1-w(\varphi))(a\cos^2\theta + b(\varphi)\sin^2\theta) + w(\varphi)\sqrt{a^2\cos^4\theta + 2d(\varphi)a\cos^2\theta\sin^2\theta + e^2(\varphi)\sin^4\theta}, \quad (1)$$

where θ and φ are the polar and azimuthal angles of phase propagation direction, measured from the positive z-axis ($0 \le \theta \le \pi$) and in a positive sense from the x-axis ($0 \le \varphi \le 2\pi$), respectively; $w(\varphi)$ is an azimuth-dependent weight used to link the elliptic and anelliptic parts of ν_{P}^2; a is the phase velocity squared of vertically propagating P-waves; $b(\varphi)$, $d(\varphi)$, and $e(\varphi)$ are functions of the azimuth φ. Equation 1 is called the GMA-type approximation because it has a form similar to the GMA (Fomel and Stovas, 2010; Stovas, 2010; Stovas and Fomel, 2012). The Taylor expansions of the exact P-wave velocity squared with respect to θ approximately $\theta = 0°$ and 90° are used to determine all parameters in equation 1. The exact phase velocity squared of P-waves is expanded up to the fourth order with respect to the polar angle θ, approximately $\theta = 0$:

$$\nu_{\mathrm{P}}^2(\theta,\varphi) = m_0 + m_2(\varphi)\theta^2 + m_4(\varphi)\theta^4, \tag{2}$$

where m_i, $i = 0,2,4$ denote the series coefficients. These coefficients are defined in the first section of Appendix C. The second-order expansion along the horizontal direction is given by

$$\nu_{\mathrm{P}}^2(\theta,\varphi) = n_0(\varphi) + n_2(\varphi)(\theta - \pi/2)^2, \tag{3}$$

where the azimuth-dependent series coefficients $n_0(\varphi)$ and $n_2(\varphi)$ are defined in the second section of Appendix C. The series coefficients in equations 2 and 3 are analytically represented in terms of medium parameters. For an elastic orthorhombic medium, the medium parameters are the density-normalized stiffness coefficients c_{ij}; for an acoustic orthorhombic medium, the medium parameters are defined in Appendix B. By matching series expansions 2 and 3 with the corresponding expansions of the GMA-type approximation 1, we determine all parameters in the GMA-type approximation 1,

$$a = m_0, \tag{4}$$

$$b(\varphi) = m_0\frac{b_1}{b_2}, \tag{5}$$

$$b_1(\varphi) = 3n_0^3 - (3m_0 + m_2 - 6m_4)n_0^2 + 2(m_2 + 3m_4)n_0n_2 - (3m_0^2 + 8m_0m_2 + 3m_2^2 + 6m_0m_4)n_0 + 3(m_0 + m_2)^3, \tag{6}$$

$$\begin{aligned} b_2(\varphi) = {} & 3m_0^3 + m_0^2(4m_2 - 6m_4 - 3n_0 + 3n_2) \\ & + m_0(3m_2^2 + 6m_4(n_0 + n_2) \\ & -3n_0(n_0 + 2n_2) + 2m_2(n_0 + 4n_2)) \\ & + 3(m_2 - n_0)^2(n_0 + n_2), \end{aligned} \tag{7}$$

$$d(\varphi) = \frac{2m_0(m_2 + 3m_4)}{3(n_0 - m_0 - m_2)} + \frac{m_0(n_0 - m_0 - m_2)}{n_0 + n_2 - m_0}, \tag{8}$$

$$e(\varphi) = -\frac{m_0(n_0 - m_0 - m_2)}{n_0 + n_2 - m_0}, \tag{9}$$

and

$$w(\varphi) = \frac{3(m_0 + m_2 - n_0)^2(m_0 - n_0 - n_2)}{2m_0(3m_0^2 + 3m_2^2 + m_0(5m_2 - 3(m_4 + 2n_0)) + m_2(-5n_0 + n_2) + 3(n_0^2 + m_4(n_0 + n_2)))}. \tag{10}$$

The argument φ is omitted for functions $b_1(\varphi)$, $b_2(\varphi)$, $m_2(\varphi)$, $m_4(\varphi)$, $n_0(\varphi)$, and $n_2(\varphi)$ in the right sides of equations 5–10.

The Fomel approximation

A four-parameter anelliptic approximation is proposed by Fomel (2004) for phase and group velocities of P-waves in a TI medium. By analogy with this approximation, the Fomel approximation for phase velocity of P-waves in an orthorhombic medium is defined as

$$\begin{aligned} \nu_{\mathrm{P}}^2(\theta,\varphi) & \\ = {} & (1 - s(\varphi))(a\cos^2\theta + c(\varphi)\sin^2\theta) \\ & + s(\varphi)\sqrt{(a\cos^2\theta + c(\varphi)\sin^2\theta)^2 + 2\frac{f(\varphi)}{s(\varphi)}\cos^2\theta\sin^2\theta}, \end{aligned} \tag{11}$$

where the parameter a is the phase velocity squared of vertically propagating P-waves, and functions $c(\varphi)$, $f(\varphi)$, and $s(\varphi)$ can be obtained in a similar way as we used for the GMA-type approximation 1. We match the fourth-order expansions of equation 11 and the exact phase velocity squared along the vertical direction. The only difference is that here we match the approximate phase velocity from equation 11 and the exact phase velocity in the horizontal direction. In other words, the second-order coefficient n_2 in equation 3 is not used. Consequently, the parameters in the Fomel approximation 11 are given by

$$c(\varphi) = n_0, \tag{12}$$

$$f(\varphi) = m_0(m_0 + m_2 - n_0), \tag{13}$$

and

$$s(\varphi) = -\frac{3(m_0 + m_2 - n_0)^2}{6(m_2 - n_0)n_0 + 2m_0(m_2 + 3(m_4 + n_0))}, \tag{14}$$

and the parameter a is given by equation 4. The argument φ of functions $m_2(\varphi)$, $m_4(\varphi)$, and $n_0(\varphi)$ is omitted in the right sides of equations 12–14.

The simplified Fomel approximation for acoustic orthorhombic media

The Fomel approximation discussed above can be further simplified for acoustic orthorhombic media. The azimuthal weight function $s(\varphi)$ in equation 11 is found to be insensitive to the azimuth φ by numerical experiments and can be approximated by a constant $s(\varphi) \approx 1/2$. To illustrate this idea, Figure 1 shows an example of the variation in the weight $s(\varphi)$ versus the azimuth φ. From this figure, we can see that the deviation of the weight $s(\varphi)$ from one half is very small. For acoustic VTI media, the weight $s(\varphi)$ becomes one half (Fomel, 2004). Therefore, the weight $s(\varphi)$ in equation 11 is assumed to be a constant

$s(\varphi) \equiv 1/2$. Thus, we obtain the simplified version of the Fomel approximation for acoustic orthorhombic media,

$$v_{\mathrm{P}}^2(\theta,\varphi) = \frac{1}{2}v_{\mathrm{P0}}^2(\cos^2\theta + \alpha(\varphi)\sin^2\theta) + \frac{1}{2}v_{\mathrm{P0}}^2\sqrt{(\cos^2\theta + \alpha(\varphi)\sin^2\theta)^2 + 4\beta(\varphi)\cos^2\theta\sin^2\theta}, \tag{15}$$

with

$$\alpha(\varphi) = \frac{1}{2}(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi) + \frac{1}{2}\sqrt{(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi)^2 + \frac{1}{\xi_3^2}r_1r_2\xi_1^2\xi_2^2\sin^2(2\varphi)}, \tag{16}$$

and

$$\beta(\varphi) = r_1\sin^2\varphi + r_2\cos^2\varphi - \alpha(\varphi), \tag{17}$$

where r_1, r_2, ξ_1, ξ_2, and ξ_3 are parameters defined in the modified Alkhalifah notation (Appendix B). This new approximation is far simpler than the GMA-type approximation 1 and the Fomel approximation 11.

ANELLIPTIC APPROXIMATIONS FOR GROUP VELOCITY

The GMA-type anelliptic approximation

The GMA-type group velocity approximation has the following form:

$$\frac{1}{v_{\mathrm{P}}^2(\Theta,\Phi)} = (1-W(\Phi))(A\cos^2\Theta + B(\Phi)\sin^2\Theta) + W(\Phi)\sqrt{A^2\cos^4\Theta + 2D(\Phi)A(\Phi)\cos^2\Theta\sin^2\Theta + E^2(\Phi)\sin^4\Theta}. \tag{18}$$

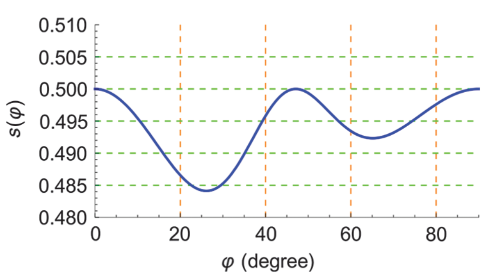

Figure 1. The variation in weight function $s(\varphi)$ from the Fomel approximation versus the azimuth φ of phase propagation direction for an acoustic orthorhombic medium. The medium parameters in Tsvankin's notation are $v_{\mathrm{P0}} = 3.0$ km/s, $\varepsilon^{(1)} = 0.25$, $\delta^{(1)} = 0.05$, $\varepsilon^{(2)} = 0.15$, $\delta^{(2)} = -0.05$, and $\delta^{(3)} = -0.1$.

The form of this approximation is the 3D extension of the 2D group velocity formula transformed from the GMA (Fomel and Stovas, 2010; Stovas, 2010). Here, Θ and Φ denote the polar and azimuthal angles of group propagation direction, measured from the positive z-axis $(0 \le \Theta \le \pi)$ and in a positive sense from the x-axis $(0 \le \Phi \le 2\pi)$, respectively; $W(\Phi)$ is the azimuth-dependent weight; A is the inverse of P-wave vertical velocity squared; $B(\Phi)$, $D(\Phi)$, and $E(\Phi)$ are coefficients dependent on the azimuth Φ. To determine all these parameters, we adopt the same approach as for the phase velocity approximation 1 by matching the Taylor expansions of approximation 18 with the corresponding expansions of the exact group velocity squared along the vertical and horizontal directions. The inverse of the exact group velocity squared of P-waves is expanded into up to the fourth order with respect to the polar angle Θ along the z-axis:

$$\frac{1}{v_{\mathrm{P}}^2(\Theta,\Phi)} = M_0 + M_2(\Phi)\Theta^2 + M_4(\Phi)\Theta^4, \tag{19}$$

where the expressions for series coefficients M_i, $i = 0,2,4$ are derived in the first section of Appendix D. The second-order expansion along the horizontal direction yields

$$\frac{1}{v_{\mathrm{P}}^2(\Phi,\Theta)} = N_0(\Phi) + N_2(\Phi)(\Theta - \pi/2)^2, \tag{20}$$

where the expressions for series coefficients N_i, $i = 0,2$ are shown in the second and third sections of Appendix D. The parameter A and coefficients $B(\Phi)$, $D(\Phi)$, and $E(\Phi)$ in approximation 18 are determined in a similar way to that in approximation 1. Consequently, they are obtained by replacing the lowercase letters in equations 4–10 by the corresponding uppercase letters.

The Fomel approximation

Corresponding to the phase velocity approximation 11, the Fomel approximation for the P-wave group velocity is defined as

$$\frac{1}{v_{\mathrm{P}}^2(\Theta,\Phi)} = (1-S(\Phi))(A\cos^2\Theta + C(\Phi)\sin^2\Theta) + S(\Phi)\sqrt{(A\cos^2\Theta + C(\Phi)\sin^2\Theta)^2 + 2\frac{F(\Phi)}{S(\Phi)}\cos^2\Theta\sin^2\Theta}, \tag{21}$$

where A, $S(\Phi)$, $C(\Phi)$, and $F(\Phi)$ are determined by replacing the lowercase letters in equations 4, 12–14 by the corresponding uppercase ones.

NUMERICAL EXAMPLES

In this section, we compare the accuracy of the proposed approximations and existing approximations for phase and group velocities in elastic and acoustic orthorhombic media. For the P-wave phase velocity, the existing approximations used here are the Tsvankin (1997) approximation with six parameters, first-order approximation (Song et al., 2001; Daley and Krebes, 2004a, 2004b), the Farra (2001) second-order approximation with nine parameters, and the Sripanich and Fomel (2014) approximation with nine parameters. For P-wave group velocity, the existing approximations used for comparisons include the six-parameter approximation proposed by Xu et al. (2005) and Vasconcelos and Tsvankin (2006), the first-order approximation (Song and Every, 2000; Daley and Krebes, 2004b) with six parameters, the Sripanich and Fomel (2014) approximation with nine parameters, the Pšenčík and Vavrycuk (2002) approximation with six weak anisotropy parameters, and the second-order approximation (Farra and Pšenčík, 2013) with six weak anisotropy parameters and a factor (the ratio of P- and S-waves velocities in reference isotropic media). The reference isotropic media for the Pšenčík and Vavrycuk (2002) approximation and the second-order approximation (Farra and Pšenčík, 2013) are determined by the P- and S-waves velocities along the vertical axis of elastic orthorhombic media. The surfaces of phase and group velocities from all approximations mentioned above and ours are symmetric with respect to the coordinate origin and symmetry planes for an orthorhombic medium no matter whether the considered medium is elastic or acoustic. For simplicity, we will plot the velocity error for one-eighth of the velocity surface of P-waves in a considered orthorhombic medium. In other words, polar angle and azimuth of phase (group) propagation direction are taken from 0° to 90°.

To test the accuracy of the proposed approximations, we first adopt the elastic orthorhombic model used by Schoenberg and Helbig (1997) and Sripanich and Fomel (2014). Figure 2 shows the accuracy comparison of our approximations with other existing approximations for phase velocities of P-wave. From this figure, we can see that the GMA-type approximation for phase velocities is the most accurate one. Its maximum relative error in the phase velocity is less than 0.002%. We can also see that the Fomel approximation is comparable with the nine-parameter approximation proposed by Sripanich and Fomel (2014) and the second-order approximation proposed by Farra (2001), but the Fomel approximation does not behave so well as the GMA-type approximation in the two vertical symmetry planes of this orthorhombic model. Figure 3 shows the accuracy comparison for a few group velocity approximations. Similar to Figure 2, the ranges of polar angle Θ and azimuth Φ are taken from 0° to 90° because the group velocity surface has the same symmetric properties as the phase velocity surface for an orthorhombic medium. The accuracy of the GMA-type approximation for group velocity is much worse than the corresponding one for phase velocity. This is because the series coefficient N_2 in equation 20 is approximately calculated for the GMA-type approximation for group velocity. The Fomel approximation for group velocity is accurate but worse than the GMA-type approximation. In this example, we can see that the Fomel and GMA-type approximations are comparable with the Sripanich and Fomel (2014) approximation and the second-order approximation (Farra and Pšenčík, 2013).

In the second example, we adopt the acoustic orthorhombic medium used in Figure 1. From the medium parameters in Figure 1, we can calculate the parameters in the modified Alkhalifah's notation (Appendix B) to use the proposed approximations for acoustic orthorhombic models. Figures 4 and 5 show the accuracy comparisons between approximations for phase and group velocities, respectively. Figure 4 indicates that all three proposed approximations for phase velocity (including the GMA-type approximation,

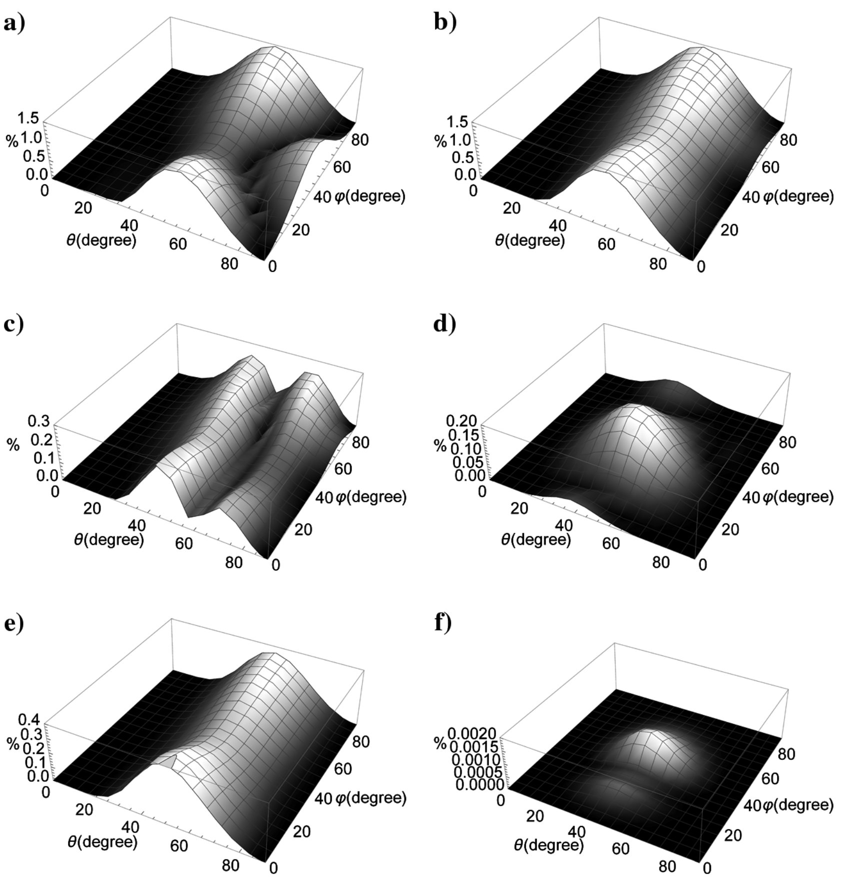

Figure 2. Relative error in P-wave phase velocity for (a) Tsvankin (1997), (b) the first-order (Song et al., 2001; Daley and Krebes, 2004a, 2004b), (c) Farra (2001) second order, (d) Sripanich and Fomel (2014), (e) the Fomel, and (f) the GMA-type approximations for an elastic orthorhombic model. The density-normalized stiffness coefficients of this model include $c_{11} = 9.0$, $c_{12} = 3.6$, $c_{13} = 2.25$, $c_{22} = 9.84$, $c_{23} = 2.4$, $c_{33} = 5.9375$, $c_{44} = 2.0$, $c_{55} = 1.6$, and $c_{66} = 2.182$, where c_{ij} have the dimension of $(\mathrm{km/s})^2$.

the Fomel approximation, and the simplified version of the Fomel approximation) are very accurate, compared with other existing approximations. From the viewpoint of practical implementations, the simplified Fomel approximation is most attractive because of its simple and compact forms. From Figure 5, we can see that the GMA-type and the Fomel approximations for group velocities are more accurate than other ones.

Next, we quantitatively calculate maximum errors of our approximations for the phase and group velocities. Table 1 shows four elastic orthorhombic models from rock physics experiments and physical models. Tables 2 and 3 illustrate the maximum relative error of our approximations and other existing approximations in phase and group velocities, respectively. From Table 2, we can see that for the phase velocities of P-waves, the GMA-type approximation is more accurate than the Fomel approximation, although the maximum errors of the GMA-type and Fomel approximations are small enough. In contrast, Table 3 indicates that for the group velocity of P-waves, the GMA-type approximation does not always behave better than the Fomel approximation. The GMA-type and Fomel approximations are very accurate, as are the Sripanich and Fomel (2014) approximation and the second-order approximation (Farra and Pšenčík, 2013) because their maximum relative errors are less than 0.4% for all given models.

In the last example, we show the maximum relative error of different approximations in the case of acoustic orthorhombic models. The density-normalized stiffness coefficients in Table 1 are converted to the Thomsen-type parameters in Tsvankin's notation (Appendix A). The corresponding acoustic models can be further obtained by neglecting the S-wave velocity parameter ν_{S0} and Thomsen-type parameters $\gamma^{(1)}$ and $\gamma^{(2)}$ in Tsvankin's notation. Table 4 lists four acoustic orthorhombic models corresponding to the elastic models shown in Table 1. Tables 5 and 6 show maximum relative errors of approximations for phase and group velocities, respectively. From Table 5, we can see that the GMA-type and Fomel approximations for phase velocity are very accurate and stable for all given acoustic orthorhombic models; the simplified version of

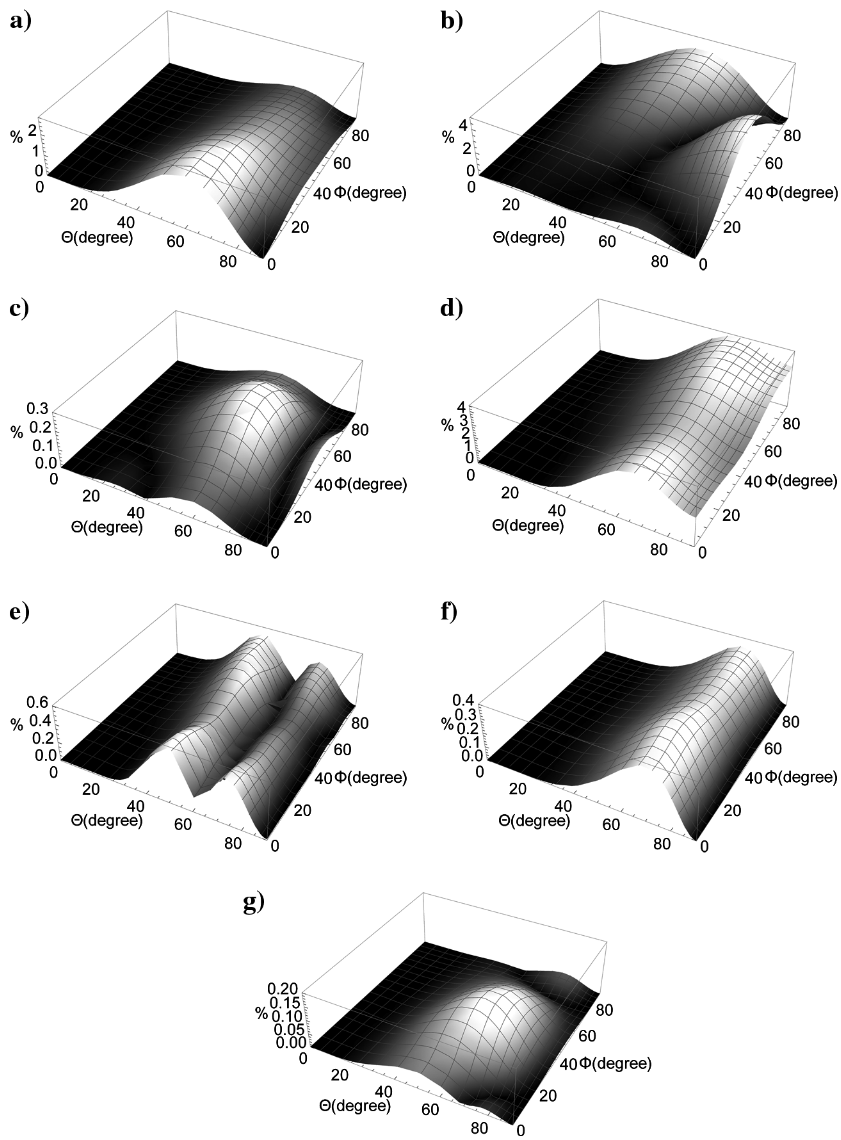

Figure 3. Relative error in P-wave group velocity for (a) Xu et al. (2005) and Vasconcelos and Tsvankin (2006), (b) the first-order (Song and Every, 2000; Daley and Krebes, 2004b), (c) Sripanich and Fomel (2014), (d) Pšenčík and Vavrycuk (2002), (e) the second-order approximation (Farra and Pšenčík, 2013), (f) the Fomel, and (g) the GMA-type approximations for an elastic orthorhombic model. The model parameters are the same as for Figure 2.

the Fomel approximation is also accurate and stable, although its accuracy is slightly lower than the Fomel approximation. From Table 6, we can see that for group velocities of P-waves, the GMA-type approximation is not always relatively accurate compared with the Fomel approximation. This phenomenon is the same as for elastic orthorhombic models shown in the previous example. It indicates that for phase and group velocities of P-waves in acoustic orthorhombic models, the Fomel approximation acts as an alternative of the GMA-type approximation.

DISCUSSION

The group velocity of a considered body wave is intrinsically linked to its moveout. The formulas for phase and group velocity approximations can be built by analogy with the forms of moveout approximations. Based on this fact, we propose the GMA-type approximation for P-wave phase and group velocities in orthorhombic media. Compared with the GMA-type approximation, the Fomel approximation does not require the second-order derivative of velocity squared with respect to polar angle for the horizontal plane of an orthorhombic medium. Although the proposed approximations are derived for orthorhombic media requiring the symmetry planes of medium to coincide with the coordinate planes, a simple coordinate rotation is enough to apply the proposed approximations for tilted orthorhombic media.

The disadvantage of the proposed approximations is obvious: First, the proposed approximations have complex forms compared with existing approximations; many parameters are used in general forms of the proposed approximations (including the GMA-type and Fomel approximations). Second, the proposed approximations have no symmetric forms with respect to exchanging the three principal axes of an orthorhombic medium. This is different from the approximation proposed by Sripanich and Fomel (2014), the first-order approximations (Song and Every, 2000; Song et al., 2001; Daley and Krebes, 2004a, 2004b) and the approximations proposed by Pšenčík and Vavrycuk (2002), and by Farra (2004). The preferred axis of the proposed approximations is the vertical axis, around which the result is relatively accurate. Finally, the proposed

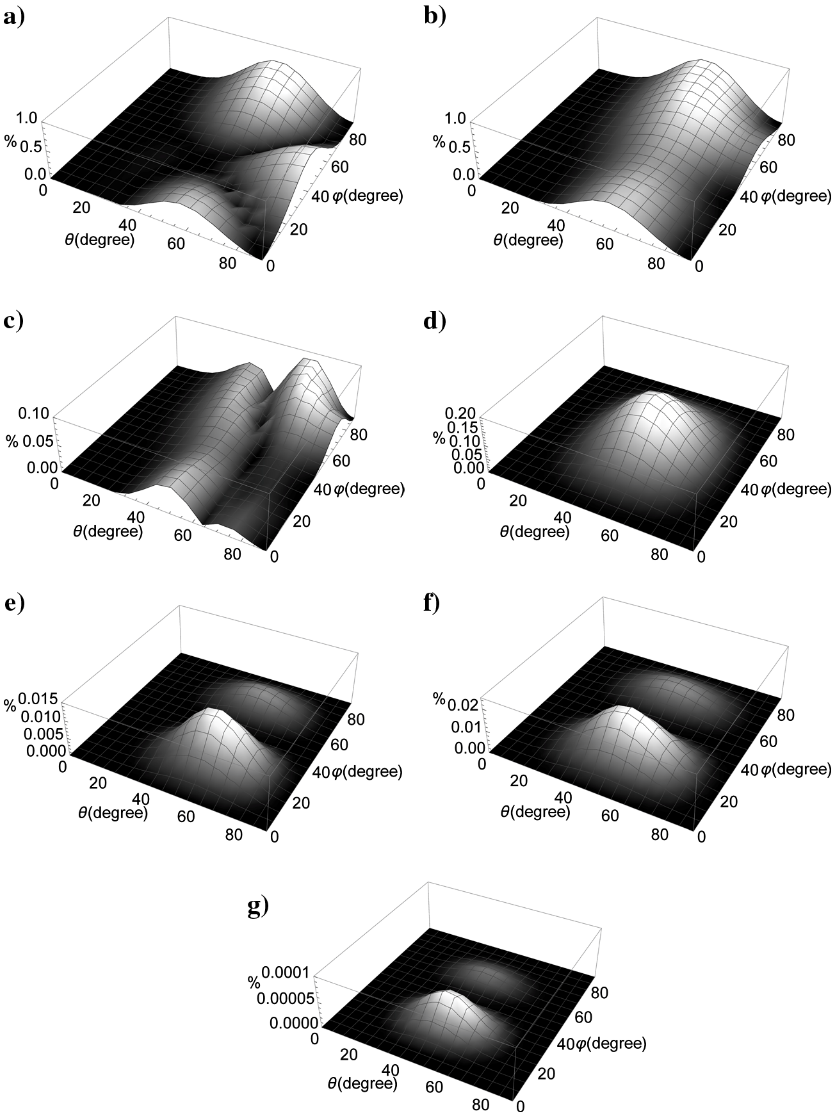

Figure 4. Relative error in P-wave phase velocity for (a) Tsvankin (1997), (b) first-order (Song et al., 2001; Daley and Krebes, 2004a, 2004b), (c) Farra (2001) second-order approximation, (d) Sripanich and Fomel (2014), (e) the Fomel, (f) the simplified Fomel, and (g) the GMA-type approximations for an acoustic orthorhombic model. The model parameters are the same as for Figure 1.

approximations cannot be easily extended to anisotropic media of lower symmetry such as monoclinic media because it is very difficult to obtain the Taylor expansions of phase velocity squared and the inverse of group velocity squared with respect to polar angles of phase and group velocity directions, to determine coefficients in the formulae for the GMA-type and Fomel approximations for that media.

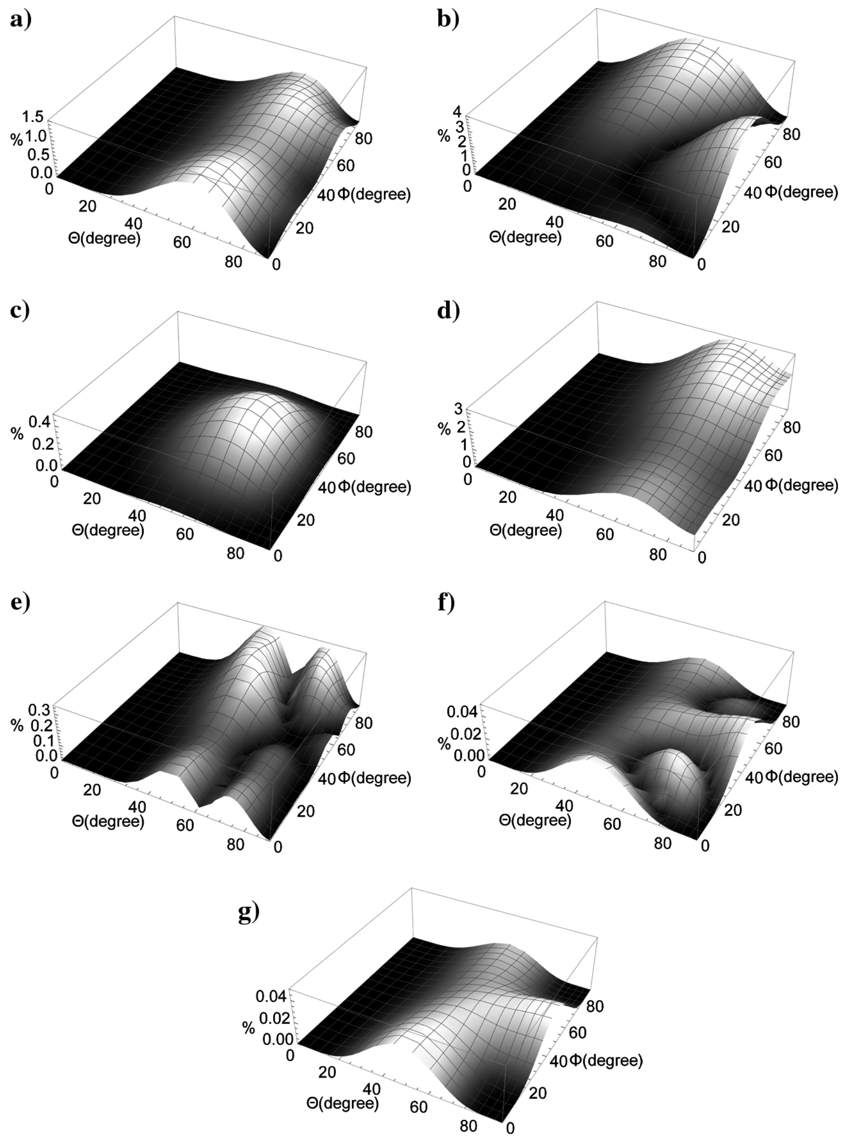

Figure 5. Similar to Figure 3 but for the acoustic orthorhombic model used in Figure 1.

Table 1. Density-normalized stiffness parameters (unit: km^2/s^2) for orthorhombic models. Models 1–4 are taken from Mah and Schmitt (2003), Mahmoudian et al. (2014), Sano et al. (1992), and Miller and Spencer (1994) respectively.

Models	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{23}
1	15.9	15.5	11.1	3.4	3.0	3.8	7.0	6.8	6.9
2	8.70	13.25	12.25	2.89	2.34	2.28	4.68	5.07	5.13
3	13.75	18.49	21.39	8.55	7.57	7.38	2.30	2.77	2.02
4	6.30	6.871	5.411	1.00	0.80	1.50	2.70	2.25	2.393

Table 2. Maximum relative error in phase velocity for different approximations for models listed in Table 1. The abbreviations S and F, Fomel, and GMA-type stand for the Sripanich and Fomel (2014), the Fomel, and GMA-type approximations, respectively; Tsvankin, First-order, and Farra stand for Tsvankin (1997) approximation, the first-order approximation (Song et al., 2001; Daley and Krebes, 2004a, 2004b), and Farra second-order approximation (Farra, 2001), respectively.

Models	Tsvankin (%)	First order (%)	Farra (%)	S and F (%)	Fomel (%)	GMA type (%)
1	0.721	0.544	0.065	0.078	0.059	3.0×10^{-3}
2	0.507	0.995	0.121	0.052	0.069	2.1×10^{-4}
3	4.198	1.090	0.203	0.207	0.209	5.3×10^{-3}
4	1.282	0.932	0.148	0.724	0.186	7.0×10^{-4}

Table 3. Maximum relative error in group velocity for different approximations for models listed in Table 1. The headings X and V, first order, second order, and Fomel stand for Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation, the first-order approximation (Song and Every, 2000; Daley and Krebes, 2004b), the second-order approximation (Farra and Pšenčík, 2013), and the Fomel approximation, respectively. The abbreviations S and F, P and V, and GMA type stand for Sripanich and Fomel (2014), Pšenčík and Vavrycuk (2002), and the GMA-type approximations, respectively.

Models	X and V (%)	First order (%)	S and F (%)	P and V (%)	Second order (%)	Fomel (%)	GMA type (%)
1	0.202	1.218	0.037	1.894	0.114	0.060	0.368
2	1.344	3.846	0.126	2.070	0.345	0.057	0.107
3	0.840	1.295	0.079	2.523	0.261	0.218	0.109
4	4.958	7.767	0.572	2.075	0.437	0.156	0.167

Table 4. Acoustic orthorhombic models converted from the elastic ones shown in Table 1.

Models	v_{P0} (km/s)	$\varepsilon^{(1)}$	$\delta^{(1)}$	$\varepsilon^{(2)}$	$\delta^{(2)}$	$\delta^{(3)}$
1	3.332	0.198	0.274	0.216	0.169	−0.077
2	3.500	0.041	−0.102	−0.145	−0.178	0.065
3	4.625	−0.068	−0.097	−0.179	−0.142	0.303
4	2.326	0.135	−0.166	0.082	−0.240	−0.089

Table 5. Similar to Table 2 but for the acoustic orthorhombic models listed in Table 4. Simplified stands for the simplified Fomel approximation.

Models	Tsvankin (%)	First order (%)	Farra (%)	S and F (%)	Fomel (%)	Simplified (%)	GMA type (%)
1	0.681	0.423	4.25×10^{-2}	2.81×10^{-2}	1.52×10^{-2}	2.10×10^{-2}	9.5×10^{-4}
2	0.367	0.789	7.61×10^{-2}	5.74×10^{-2}	3.12×10^{-2}	4.45×10^{-2}	6.3×10^{-5}
3	3.889	0.624	7.28×10^{-2}	2.17×10^{-2}	1.73×10^{-2}	2.39×10^{-2}	9.3×10^{-4}
4	1.037	0.754	9.90×10^{-2}	0.458	1.86×10^{-2}	3.04×10^{-2}	2.4×10^{-4}

Table 6. Similar to Table 3 but for the acoustic orthorhombic models listed in Table 4.

Models	X and V (%)	First order (%)	S and F (%)	P and V (%)	Second order (%)	Fomel (%)	GMA type (%)
1	0.202	1.312	0.082	1.995	0.129	0.072	0.167
2	1.546	3.792	0.131	2.283	0.374	0.076	0.021
3	0.705	0.964	0.070	2.864	0.287	0.083	0.384
4	4.785	7.651	0.754	2.219	0.446	0.311	0.311

CONCLUSIONS

The proposed approximations (including the GMA-type and Fomel approximations for phase and group velocities and the simplified Fomel approximation for phase velocity) are accurate for elastic and acoustic orthorhombic media with strong anisotropy. For the acoustic orthorhombic model, the Fomel approximation for phase velocities can be reduced to a simple and accurate formula. As we stated in the "Introduction" section, the potential applications of the proposed approximations include forward modeling and migration based on the dispersion relation and the forward traveltime calculation for seismic tomography.

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APPENDIX A

TSVANKIN'S NOTATION FOR ORTHORHOMBIC MEDIA

In this appendix, we show the definition of Thomsen-type parameters in Tsvankin's (1997) notation. Tsvankin's (1997) notation includes two velocity parameters and seven dimensionless anisotropy parameters. The definition of these parameters is as follows:

1) v_{P0} — the velocity of the vertically propagating P-wave:

$$v_{P0} \equiv \sqrt{c_{33}} \tag{A-1}$$

2) v_{S0} — the velocity of the vertically propagating S-wave polarized in the x-direction:

$$v_{S0} \equiv \sqrt{c_{55}} \tag{A-2}$$

3) $\varepsilon^{(1)}, \delta^{(1)}, \gamma^{(1)}$ — the Thomsen-type VTI parameters defined in the $[y, z]$ plane (the superscript 1 corresponds to the x-axis normal to the $[y, z]$ plane):

$$\varepsilon^{(1)} \equiv \frac{c_{22} - c_{33}}{2c_{33}}, \tag{A-3}$$

$$\delta^{(1)} \equiv \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}, \tag{A-4}$$

$$\gamma^{(1)} \equiv \frac{c_{66} - c_{55}}{2c_{55}} \tag{A-5}$$

4) $\varepsilon^{(2)}, \delta^{(2)}, \gamma^{(2)}$ — the Thomsen-type VTI parameters defined in the $[x, z]$ plane (the superscript 2 corresponds to the y-axis normal to the $[x, z]$ plane):

$$\varepsilon^{(2)} \equiv \frac{c_{11} - c_{33}}{2c_{33}}, \tag{A-6}$$

$$\delta^{(2)} \equiv \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}, \tag{A-7}$$

$$\gamma^{(2)} \equiv \frac{c_{66} - c_{44}}{2c_{44}} \tag{A-8}$$

5) $\delta^{(3)}$ — the Thomsen-type VTI parameter defined in the $[x, y]$ plane (the superscript 3 corresponds to the z-axis normal to the $[x, y]$ plane; the x-axis plays the role of the symmetry axis):

$$\delta^{(3)} \equiv \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})} \tag{A-9}$$

APPENDIX B

THE MODIFIED ALKHALIFAH'S NOTATION FOR ACOUSTIC ORTHORHOMBIC MEDIA

Alkhalifah's (2003) notation is used to describe an acoustic orthorhombic medium. For this set of media, the S-wave velocity along the principal axes is assumed to be zero. Therefore, only six independent parameters are required to describe the velocities and traveltimes of P-waves in an acoustic orthorhombic medium. The definition of parameters in Alkhalifah's (2003) notation is as follows:

1) v_{P0} — the velocity of the vertically propagating P-wave:

$$v_{P0} \equiv \sqrt{c_{33}} \tag{B-1}$$

2) $v_{NMO}^{(1)}, v_{NMO}^{(2)}$ — the NMO-velocities of P-waves defined in the $[y, z]$ and $[x, z]$ planes:

$$v_{NMO}^{(1)} \equiv \sqrt{\frac{c_{23}(c_{23} + 2c_{44}) + c_{33}c_{44}}{c_{33} - c_{44}}} \tag{B-2}$$

$$v_{NMO}^{(2)} \equiv \sqrt{\frac{c_{13}(c_{13} + 2c_{55}) + c_{33}c_{55}}{c_{33} - c_{55}}} \tag{B-3}$$

3) $\eta^{(1)}, \eta^{(2)}$ — the anellipticity parameters of P-waves defined in the $[y, z]$ and $[x, z]$ planes:

$$\eta^{(1)} \equiv \frac{c_{22}(c_{33} - c_{44})}{2c_{23}(c_{23} + 2c_{44}) + 2c_{33}c_{44}} - \frac{1}{2} \tag{B-4}$$

$$\eta^{(2)} \equiv \frac{c_{11}(c_{33}-c_{55})}{2c_{13}(c_{13}+2c_{55})+2c_{33}c_{55}} - \frac{1}{2} \quad \text{(B-5)}$$

4) $\delta^{(3)}$ — the Thomsen-type parameters defined in the $[x, y]$ plane:

$$\delta^{(3)} \equiv \frac{(c_{12}+c_{66})^2-(c_{11}-c_{66})^2}{2c_{11}(c_{11}-c_{66})} \quad \text{(B-6)}$$

To simplify the derivation of approximations for phase and group velocities for acoustic orthorhombic media, we propose the modified Alkhalifah's notation: The vertical velocity parameter ν_{P0} of P-waves is conserved, whereas the other five parameters (equations B-2 to B-6) in Alkhalifah's notation are replaced by the following new parameters:

$$\xi_1 \equiv \sqrt{1+2\eta^{(1)}} = \sqrt{\frac{c_{22}(c_{33}-c_{44})}{c_{23}^2+2c_{23}c_{44}+c_{33}c_{44}}}, \quad \text{(B-7)}$$

$$\xi_2 \equiv \sqrt{1+2\eta^{(2)}} = \sqrt{\frac{c_{11}(c_{33}-c_{55})}{c_{13}^2+2c_{13}c_{55}+c_{33}c_{55}}}, \quad \text{(B-8)}$$

$$\xi_3 \equiv \sqrt{1+2\eta^{(3)}} = \sqrt{\frac{c_{22}(c_{11}-c_{66})}{c_{12}^2+2c_{12}c_{66}+c_{11}c_{66}}}, \quad \text{(B-9)}$$

$$r_1 \equiv 1+2\delta^{(1)} = \frac{c_{23}^2+2c_{23}c_{44}+c_{33}c_{44}}{c_{33}(c_{33}-c_{44})}, \quad \text{(B-10)}$$

$$r_2 \equiv 1+2\delta^{(2)} = \frac{c_{13}^2+2c_{13}c_{55}+c_{33}c_{55}}{c_{33}(c_{33}-c_{55})}. \quad \text{(B-11)}$$

In equation B-9, $\eta^{(3)}$ denotes the anellipticity parameter defined in the $[x, y]$ plane (Grechka and Tsvankin, 1999). It can also be expressed in terms of the Thomsen-type parameters:

$$\eta^{(3)} \equiv \frac{\varepsilon^{(1)}-\varepsilon^{(2)}-\delta^{(3)}(1+2\varepsilon^{(2)})}{(1+2\varepsilon^{(2)})(1+2\delta^{(3)})}. \quad \text{(B-12)}$$

The P-wave slowness surface equation (Alkhalifah, 2003) can thus be written as

$$p_3^2 = \frac{1}{\nu_{\mathrm{P0}}^2}\frac{f_1(p_1,p_2)}{f_2(p_1,p_2)}, \quad \text{(B-13)}$$

where (p_1, p_2, p_3) denotes the phase slowness vector and functions $f_1(p_1, p_2)$ and $f_2(p_1, p_2)$ are given by

$$f_1(p_1,p_2) = (1-r_1\xi_1^2p_2^2\nu_{\mathrm{P0}}^2)(1-r_2\xi_2^2p_1^2\nu_{\mathrm{P0}}^2) - \frac{1}{\xi_3^2}r_1r_2\xi_1^2\xi_2^2p_1^2p_2^2\nu_{\mathrm{P0}}^4, \quad \text{(B-14)}$$

$$f_2(p_1,p_2) = 1+r_1(1-\xi_1^2)p_2^2\nu_{\mathrm{P0}}^2+r_2(1-\xi_2^2)p_1^2\nu_{\mathrm{P0}}^2 - r_1r_2\Omega p_1^2p_2^2\nu_{\mathrm{P0}}^4, \quad \text{(B-15)}$$

with

$$\Omega = \xi_1^2+\xi_2^2-\xi_1^2\xi_2^2+\frac{\xi_1^2\xi_2^2}{\xi_3^2}-\frac{2\xi_1\xi_2}{\xi_3}. \quad \text{(B-16)}$$

APPENDIX C

TAYLOR EXPANSIONS OF THE EXACT P-WAVE PHASE VELOCITY SQUARED

In this appendix, we compute the series coefficients of the P-wave phase velocity squared at vertical and horizontal directions of an orthorhombic medium.

The Taylor expansion along the z-axis

The exact solution for the P-wave phase velocity in an elastic orthorhombic medium can be found in Every (1980) and Schoenberg and Helbig (1997). The P-wave phase velocity squared along the vertical direction is expanded with respect to the polar angle θ of phase propagation direction up to the fourth order:

$$\nu_{\mathrm{P}}^2(\varphi,\theta) = m_0+m_2(\varphi)\theta^2+m_4(\varphi)\theta^4, \quad \text{(C-1)}$$

where the series coefficients m_i, $i = 0,2,4$, are

$$m_0 = c_{33}, \quad \text{(C-2)}$$

$$m_2 = -c_{33}+\frac{(c_{13}^2+2c_{13}c_{55}+c_{33}c_{55})}{c_{33}-c_{55}}\cos^2\varphi + \frac{(c_{23}^2+2c_{23}c_{44}+c_{33}c_{44})}{c_{33}-c_{44}}\sin^2\varphi, \quad \text{(C-3)}$$

$$m_4 = m_{4xx}\cos^4\varphi+m_{4xy}\cos^2\varphi\sin^2\varphi+m_{4yy}\sin^4\varphi, \quad \text{(C-4)}$$

with

$$m_{4xx} = -\frac{1}{3(c_{33}-c_{55})^3}(3c_{13}^4-3c_{11}c_{13}^2c_{33}+c_{13}^2c_{33}^2-c_{33}^4 + (3c_{13}^2(c_{11}+4c_{13})+c_{13}(-6c_{11}+c_{13})c_{33} + 2c_{13}c_{33}^2+4c_{33}^3)c_{55} + (2c_{13}-c_{33})(3c_{11}+8c_{13}+5c_{33})c_{55}^2 + (3c_{11}+8c_{13}+5c_{33})c_{55}^3), \quad \text{(C-5)}$$

$$m_{4xy}=\frac{1}{3(c_{33}-c_{44})^2(c_{33}-c_{55})^2}(-c_{23}^2c_{33}^3+2c_{33}^5+c_{23}^2c_{33}^2c_{44}$$
$$-5c_{33}^4c_{44}+3c_{33}^3c_{44}^2+2c_{23}c_{33}^2c_{44}(-c_{33}+c_{44})$$
$$-c_{23}^2c_{33}^2c_{55}-5c_{33}^4c_{55}-2c_{23}^2c_{33}c_{44}c_{55}+6c_{12}c_{33}^2c_{44}c_{55}$$
$$+12c_{33}^3c_{44}c_{55}-6c_{12}c_{33}c_{44}^2c_{55}-10c_{33}^2c_{44}^2c_{55}$$
$$-2c_{23}c_{33}(3c_{12}(-c_{33}+c_{44})+c_{44}(c_{33}+2c_{44}))c_{55}$$
$$-c_{23}^2c_{33}c_{55}^2+3c_{33}^3c_{55}^2+4c_{23}^2c_{44}c_{55}^2-6c_{12}c_{33}c_{44}c_{55}^2$$
$$-10c_{33}^2c_{44}c_{55}^2+6c_{12}c_{44}^2c_{55}^2+10c_{33}c_{44}^2c_{55}^2$$
$$-2c_{23}(3c_{12}(c_{33}-c_{44})+(c_{33}-4c_{44})c_{44})c_{55}^2$$
$$+3(c_{33}(c_{23}+c_{44})+(-c_{23}+c_{33}-2c_{44})c_{55})^2c_{66}$$
$$+2c_{13}(3c_{12}(c_{33}-c_{44})(c_{23}+c_{44})(c_{33}-c_{55})$$
$$+c_{55}(-c_{33}(c_{33}^2+c_{33}c_{44}+c_{44}^2)+(c_{33}^2-2c_{33}c_{44}+4c_{44}^2)c_{55}$$
$$+3c_{23}^2(-2c_{33}+c_{44}+c_{55})+6c_{23}c_{44}(-2c_{33}+c_{44}+c_{55}))$$
$$+3(c_{33}-c_{44})(c_{33}(c_{23}+c_{44})+(-c_{23}+c_{33}-2c_{44})c_{55})c_{66})$$
$$+c_{13}^2(-c_{33}^3+3c_{23}^2(-2c_{33}+c_{44}+c_{55})$$
$$+6c_{23}c_{44}(-2c_{33}+c_{44}+c_{55})+c_{33}^2(-c_{44}+c_{55}+3c_{66})$$
$$+c_{44}^2(4c_{55}+3c_{66})-c_{33}c_{44}(c_{44}+2c_{55}+6c_{66}))),\quad \text{(C-6)}$$

$$m_{4yy}=-\frac{1}{3(c_{33}-c_{44})^3}(3c_{23}^4-3c_{22}c_{23}^2c_{33}$$
$$+c_{23}^2c_{33}^2-c_{33}^4+(3c_{23}^2(c_{22}+4c_{23})$$
$$+c_{23}(-6c_{22}+c_{23})c_{33}+2c_{23}c_{33}^2+4c_{33}^3)c_{44}$$
$$+(2c_{23}-c_{33})(3c_{22}+8c_{23}+5c_{33})c_{44}^2$$
$$+(3c_{22}+8c_{23}+5c_{33})c_{44}^3).\quad \text{(C-7)}$$

The number of independent stiffness parameters reduces to six for acoustic orthorhombic media. The modified Alkhalifah's notation (Appendix B) is adopted to describe the P-wave phase velocity. Consequently, the coefficients C-2 and C-3 and coefficients C-5 to C-7 become

$$m_0=\nu_{P0}^2,\quad \text{(C-8)}$$

$$m_2=\nu_{P0}^2(-1+r_2\cos^2\varphi+r_1\sin^2\varphi),\quad \text{(C-9)}$$

$$m_{4xx}=\frac{1}{3}\nu_{P0}^2(1-r_2-3r_2^2(1-\xi_2^2)),\quad \text{(C-10)}$$

$$m_{4xy}=\frac{1}{3\xi_3}\nu_{P0}^2(6r_1r_2\xi_1\xi_2+(2-r_1-r_2-6r_1r_2)\xi_3),\quad \text{(C-11)}$$

$$m_{4yy}=\frac{1}{3}\nu_{P0}^2(1-r_1-3r_1^2(1-\xi_1^2)).\quad \text{(C-12)}$$

The Taylor expansion along the horizontal direction

For the P-wave phase velocity in a vertical plane with the azimuth φ, we define its Taylor expansion as

$$\nu_P^2(\theta,\varphi)=n_0(\varphi)+n_2(\varphi)(\theta-\pi/2)^2,\quad \text{(C-13)}$$

where $n_0(\varphi)$ corresponds to the squared magnitude of the phase velocity defined in the $[x, y]$ plane. Because all three symmetry planes of an orthorhombic media exhibit transverse isotropy, $n_0(\varphi)$ is obtained by substituting the appropriate density-normalized stiffness coefficients into the phase velocity formula of P-waves in VTI media (Tsvankin, 1997):

$$n_0(\varphi)=\frac{1}{2}((c_{11}+c_{66})\cos^2\varphi+(c_{22}+c_{66})\sin^2\varphi)$$
$$+\frac{1}{2}\sqrt{((c_{11}-c_{66})\cos^2\varphi-(c_{22}-c_{66})\sin^2\varphi)^2+(c_{12}+c_{66})^2\sin^2(2\varphi)}.\quad \text{(C-14)}$$

The coefficient $n_2(\varphi)$ corresponds to the second-order derivative of the squared phase velocity evaluated at $\theta=\pi/2$. This coefficient can be determined with the aid of the implicit function theory. By taking the second-order derivative of the polynomial equation (Appendix A in Tsvankin, 1997) for the phase velocity with respect to θ at $\pi/2$, we find the expression for $n_2(\varphi)$:

$$n_2(\varphi)=\frac{-2n_0^3+\chi_2(\varphi)n_0^2(\varphi)+\chi_1(\varphi)n_0(\varphi)+\chi_0(\varphi)}{((c_{11}+c_{66})\cos^2\varphi+(c_{22}+c_{66})\sin^2\varphi-2n_0)(c_{55}\cos^2\varphi+c_{44}\sin^2\varphi-n_0(\varphi))},\quad \text{(C-15)}$$

with

$$\chi_0(\varphi)=(c_{11}c_{44}c_{55}-c_{13}(c_{13}+2c_{55})c_{66})\cos^4\varphi$$
$$+(c_{22}c_{44}c_{55}-c_{23}(c_{23}+2c_{44})c_{66})\sin^4\varphi$$
$$+(-c_{13}^2c_{22}-c_{11}c_{23}(c_{23}+2c_{44})$$
$$+2c_{55}(c_{12}(c_{23}+c_{44})+(c_{23}+2c_{44})c_{66})$$
$$+2c_{13}(-c_{22}c_{55}+(c_{23}+c_{44})(c_{12}+c_{66})))\cos^2\varphi\sin^2\varphi,\quad \text{(C-16)}$$

$$\chi_1(\varphi)=(c_{13}^2+2c_{13}c_{55}-c_{11}(c_{44}+c_{55})$$
$$-c_{55}(c_{44}+2c_{66}))\cos^4\varphi$$
$$+(c_{23}^2+2c_{23}c_{44}-c_{22}(c_{44}+c_{55})$$
$$-c_{44}(c_{55}+2c_{66}))\sin^4\varphi$$
$$+(c_{13}^2+c_{23}^2+2c_{23}c_{44}+2c_{13}c_{55}$$
$$-2(c_{11}c_{44}+c_{55}(c_{22}+c_{44})$$
$$+c_{66}(c_{44}+c_{55}))\cos^4\varphi\sin^4\varphi,\quad \text{(C-17)}$$

$$\chi_2(\varphi)=c_{44}+c_{55}+c_{66}+(c_{11}+2c_{55})\cos^2\varphi$$
$$+(c_{22}+2c_{44})\sin^2\varphi.\quad \text{(C-18)}$$

For an acoustic orthorhombic medium, coefficients $n_0(\varphi)$ and $n_2(\varphi)$ are written in the modified Alkhalifah's notation (Appendix B),

$$n_0(\varphi) = \frac{1}{2}\nu_{P0}^2(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi) + \frac{1}{2}\nu_{P0}^2\sqrt{(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi)^2 + \frac{1}{\xi_3^2}r_1r_2\xi_1^2\xi_2^2\sin^2(2\varphi)}, \quad \text{(C-19)}$$

$$n_2(\varphi) = \frac{-2n_0^3(\varphi) + \chi_2(\varphi)n_0^2(\varphi) + \chi_1(\varphi)n_0(\varphi) + \chi_0(\varphi)}{n_0(\varphi)(2n_0(\varphi) - \nu_{P0}^2(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi))}, \quad \text{(C-20)}$$

with

$$\chi_0(\varphi) = \frac{1}{\xi_3}r_1r_2\nu_{P0}^6(2\xi_1\xi_2 - \xi_1^2\xi_3 - \xi_2^2\xi_3)\cos^2\varphi\sin^2\varphi, \quad \text{(C-21)}$$

$$\chi_1(\varphi) = \nu_{P0}^4(r_2\cos^2\varphi + r_1\sin^2\varphi), \quad \text{(C-22)}$$

$$\chi_2(\varphi) = \nu_{P0}^2(r_2\xi_2^2\cos^2\varphi + r_1\xi_1^2\sin^2\varphi). \quad \text{(C-23)}$$

APPENDIX D

TAYLOR EXPANSIONS OF THE INVERSE OF THE EXACT P-WAVE GROUP VELOCITY SQUARED

In this appendix, we show the Taylor expansions of the inverse of the group velocity squared of P-waves at vertical and horizontal directions.

The Taylor expansion along the z-axis

Al-Dajani and Toksoz (2002) derive the P-wave moveout expansion for a horizontal orthorhombic layer. From their result, we find the fourth-order Taylor polynomial for the inverse of the group velocity squared of P-waves with respect to the polar angle Θ of group propagation direction at the z-axis:

$$\frac{1}{\nu_P^2(\Theta,\Phi)} = M_0 + M_2(\Phi)\Theta^2 + M_4(\Phi)\Theta^4, \quad \text{(D-1)}$$

where the series coefficients M_i, $i = 0,2,4$ are given by

$$M_0 = \frac{1}{c_{33}}, \quad \text{(D-2)}$$

$$M_2(\Phi) = -\frac{1}{c_{33}} + \frac{c_{33}-c_{55}}{c_{13}^2 + 2c_{13}c_{55} + c_{33}c_{55}}\cos^2\Phi + \frac{c_{33}-c_{44}}{c_{23}^2 + 2c_{23}c_{44} + c_{33}c_{44}}\sin^2\Phi, \quad \text{(D-3)}$$

$$M_4(\Phi) = \frac{(c_{13}+c_{33})(c_{13}-c_{33}+2c_{55})}{3c_{33}(c_{13}^2+2c_{13}c_{55}+c_{33}c_{55})}\cos^4\Phi + \frac{(c_{23}+c_{33})(c_{23}-c_{33}+2c_{44})}{3c_{33}(c_{23}^2+2c_{23}c_{44}+c_{33}c_{44})}\sin^4\Phi + \frac{1}{3}\left(\frac{2}{c_{33}} - \frac{(c_{33}-c_{44})}{c_{23}^2+2c_{23}c_{44}+c_{33}c_{44}} - \frac{(c_{33}-c_{55})}{c_{13}^2+2c_{13}c_{55}+c_{33}c_{55}}\right)\cos^2\Phi\sin^2\Phi. \quad \text{(D-4)}$$

For an acoustic orthorhombic medium, we adopt the modified Alkhalifah's notation (Appendix B) to represent coefficients D-2 to D-4. Consequently, these coefficients are reduced to

$$M_0 = \frac{1}{\nu_{P0}^2}, \quad \text{(D-5)}$$

$$M_2(\Phi) = \frac{1}{\nu_{P0}^2}\left(-1 + \frac{1}{r_2}\cos^2\Phi + \frac{1}{r_1}\sin^2\Phi\right), \quad \text{(D-6)}$$

$$M_4(\Phi) = \frac{1}{\nu_{P0}^2}\left(\frac{r_2-1}{3r_2}\cos^4\Phi - \frac{r_1+r_2-2r_1r_2}{3r_1r_2}\cos^2\Phi\sin^2\Phi + \frac{r_1-1}{3r_1}\sin^4\Phi\right). \quad \text{(D-7)}$$

The Taylor expansion along the horizontal direction

The second-order Taylor expansion of the inverse of P-wave group velocity squared along the horizontal direction is defined as

$$\frac{1}{\nu_P^2(\Phi,\Theta)} = N_0(\Phi) + N_2(\Phi)(\Theta - \pi/2)^2 + \cdots, \quad \text{(D-8)}$$

where the zero-order coefficient $N_0(\Phi)$ corresponds to the inverse of the P-wave group velocity defined in [x, y] plane. Because the [x, y] plane, as a symmetry plane of an orthorhombic medium, displays transverse isotropy, the P-wave GMA for 2D VTI media (Stovas, 2010) is rewritten as the approximation for $N_0(\Phi)$ by the appropriate substitution of density-normalized stiffness coefficients,

$$N_0(\Phi) = (1-\zeta)\left(\frac{\cos^2\Phi}{c_{11}} + \frac{\sin^2\Phi}{a_{SF}}\right) + \zeta\sqrt{\frac{\cos^4\Phi}{c_{11}^2} + \frac{2\cos^2\Phi\sin^2\Phi}{c_{11}b_{SF}} + \frac{\sin^4\Phi}{c_{SF}^2}}, \quad \text{(D-9)}$$

with

$$\zeta = \frac{(c_{12}^2 + c_{11}c_{66} + 2c_{12}c_{66})^2}{2(c_{12}^4 + 4c_{12}^3c_{66} + c_{11}c_{22}c_{66}^2 + 2c_{11}c_{12}c_{66}(c_{22}+c_{66}) + c_{12}^2(4c_{66}^2 + c_{11}(c_{22}+c_{66})))}, \quad \text{(D-10)}$$

$$a_{\mathrm{SF}} = \frac{c_{22}(c_{12}^4 + 4c_{12}^3 c_{66} + 4c_{11}c_{12}c_{22}c_{66} + c_{11}(-c_{11} + 2c_{22})c_{66}^2 + 2c_{12}^2(c_{11}c_{22} + 2c_{66}^2))}{c_{12}^4 + 4c_{12}^3 c_{66} + c_{11}c_{22}c_{66}^2 + 2c_{12}c_{66}(2c_{11}c_{22} + c_{11}c_{66} - c_{22}c_{66}) + c_{12}^2(c_{22}(2c_{11} - c_{66}) + c_{66}(c_{11} + 4c_{66}))}, \tag{D-11}$$

$$\begin{aligned} \frac{1}{b_{\mathrm{SF}}} = -&\frac{1}{c_{22}(c_{12}^2 + c_{11}c_{66} + 2c_{12}c_{66})^3} \\ &\times (c_{12}^6 + 6c_{12}^5 c_{66} + 4c_{12}^3 c_{66}^2(2c_{11} + c_{22} + 2c_{66}) \\ &+ c_{12}^4 c_{66}(2c_{11} + c_{22} + 12c_{66}) \\ &+ c_{11}c_{22}c_{66}^2(2c_{22}c_{66} + c_{11}(-2c_{22} + c_{66})) \\ &+ 2c_{11}c_{12}c_{66}(2c_{22}c_{66}(c_{22} + c_{66}) \\ &+ c_{11}(-2c_{22}^2 + c_{66}^2)) + c_{12}^2(4c_{22}c_{66}^3 + c_{11}^2(-2c_{22}^2 + c_{66}^2) \\ &+ 2c_{11}c_{66}(c_{22}^2 + c_{22}c_{66} + 4c_{66}^2))), \end{aligned} \tag{D-12}$$

$$c_{\mathrm{SF}} = \frac{c_{22}(c_{12}^2 + c_{11}c_{66} + 2c_{12}c_{66})}{c_{12}^2 + 2c_{12}c_{66} + c_{22}c_{66}}. \tag{D-13}$$

For acoustic orthorhombic media, the stiffness coefficient c_{11}, in equation D-9, is expressed by, in the modified Alkhalifah notation (Appendix B),

$$c_{11} = r_2 \xi_2^2 \nu_{\mathrm{P0}}^2, \tag{D-14}$$

and equations D-10 to D-13 become

$$\zeta = \frac{1}{2(1 + \xi_3^2)}, \tag{D-15}$$

$$a_{\mathrm{SF}} = c_{\mathrm{SF}} = r_1 \xi_1^2 \nu_{\mathrm{P0}}^2, \tag{D-16}$$

$$b_{\mathrm{SF}} = \frac{r_1 \xi_1^2 \nu_{\mathrm{P0}}^2}{-1 + 2\xi_3^4}. \tag{D-17}$$

To determine the second-order coefficient $N_2(\Phi)$ in equation D-8, we consider the reflection traveltime of P-waves in a horizontal orthorhombic layer with the fixed depth z. For the given acquisition azimuth Φ, the asymptotic expansion of the two-way traveltime squared at the infinite offset is assumed to behave as

$$t_r^2 \approx t_\infty^2 + p_\infty^2 h_r^2, \tag{D-18}$$

by analogy with the asymptotic expansion for a horizontal VTI layer (Fomel and Stovas, 2010; Stovas, 2010). In equation D-18, h_r denotes the source-receiver offset, t_r denotes the two-way traveltime, t_∞ denotes the intercept of the asymptotic traveltime, and $p_\infty = \sqrt{N_0(\Phi)}$ denotes the group slowness of P-waves defined in the $[x, y]$ plane, where $N_0(\Phi)$ is given by equation D-9. According to Stovas (2010), the asymptotic traveltime intercept t_∞ is determined by

$$t_\infty^2 = \lim_{\theta \to \pi/2} (t_r^2 - p_r t_r h_r), \tag{D-19}$$

where θ denotes the polar angle of the phase propagation direction of incident P-waves, measured from the z-axis; p_r denotes the projection of the slowness vector on the acquisition azimuth, corresponding to the azimuth-dependent slope of the t_r-h_r curve.

To relate the traveltime t_r to the group velocity ν_{P}, we take into account the P-wave propagation distance given by

$$R = \sqrt{h_r^2 + (2z)^2}. \tag{D-20}$$

Dividing equation D-18 by the square of equation D-20 yields the asymptotic expansion of the inverse of the group velocity squared as the polar angle Θ of group propagation direction approaches $\pi/2$:

$$\frac{1}{\nu_{\mathrm{P}}^2(\Theta, \Phi)} \approx p_\nu^2 \cos^2\Theta + p_\infty^2 \sin^2\Theta, \tag{D-21}$$

where p_∞ is explained after equation D-18; p_ν is defined as $p_\nu^2 \equiv t_\infty^2/(2z)^2$, which is obtained from equation D-19:

$$p_\nu^2 = \frac{1}{(2z)^2} \lim_{\theta \to \pi/2} (t_r^2 - p_r t_r h_r). \tag{D-22}$$

The second-order Taylor expansion of equation D-21 at $\Theta = \pi/2$ is given by

$$\frac{1}{\nu_{\mathrm{P}}^2(\Theta, \Phi)} = p_\infty^2 + (p_\nu^2 - p_\infty^2)\left(\Theta - \frac{\pi}{2}\right)^2. \tag{D-23}$$

Matching equation D-8 with equation D-23 allows us to find the second-order coefficient $N_2(\Phi)$ defined in equation D-8:

$$N_2(\Phi) = \frac{\Omega(\Phi)}{c_{33}} - N_0(\Phi), \tag{D-24}$$

where $\Omega(\Phi)$ is given by

$$\Omega(\Phi) = \frac{1}{t_0^2} \lim_{\theta \to \pi/2} (t_r^2 - p_r t_r h_r). \tag{D-25}$$

Here, $t_0 = 2z/\sqrt{c_{33}}$ denotes the zero-offset two-way traveltime, and $\Omega(\Phi)$ is approximately decomposed into the following two terms:

$$\Omega(\Phi) = \Omega_{\mathrm{acoustic}}(\Phi) + \Omega_{\mathrm{elastic}}(\Phi), \tag{D-26}$$

where the term $\Omega_{\mathrm{acoustic}}$ corresponds to the result of $\Omega(\Phi)$ for acoustic orthorhombic media; the term $\Omega_{\mathrm{elastic}}$ denotes the perturbation of $\Omega(\Phi)$ from acoustic to elastic orthorhombic media; the term $\Omega_{\mathrm{elastic}}$ will vanish for acoustic orthorhombic media. The terms $\Omega_{\mathrm{acoustic}}$ and $\Omega_{\mathrm{elastic}}$ are derived in the next section of this appendix.

For acoustic orthorhombic media, the series coefficients N_0 and $N_2(\Phi)$ given by equations D-9 and D-24 are reduced to

$$\begin{aligned} N_0(\Phi) = &\frac{1 + 2\xi_3^2}{2(1 + \xi_3^2)\nu_{\mathrm{P0}}^2}\left(\frac{\cos^2\Phi}{r_2\xi_2^2} + \frac{\sin^2\Phi}{r_1\xi_1^2}\right) \\ &+ \frac{1}{2(1 + \xi_3^2)\nu_{\mathrm{P0}}^2}\sqrt{\frac{\cos^4\Phi}{(r_2\xi_2^2)^2} - 2(1 - 2\xi_3^4)\frac{\cos^2\Phi \sin^2\Phi}{r_1 r_2 \xi_1^2 \xi_2^2} + \frac{\sin^4\Phi}{(r_1\xi_1^2)^2}}, \end{aligned} \tag{D-27}$$

$$N_2(\Phi) = \frac{\Omega_{\text{acoustic}}(\Phi)}{\nu_{\text{P0}}^2} - N_0(\Phi). \quad \text{(D-28)}$$

The derivation of Ω defined in equation D-25

For simplicity, we first consider the case of acoustic orthorhombic media. The modified Alkhalifah's notation (equation B-1 and equations B-7 to B-11) is adopted to parameterize such media. The traveltime t_r in equation D-25 is calculated by the following equation:

$$t_r = p_1 x + p_2 y + 2p_3 z, \quad \text{(D-29)}$$

with

$$x = -2z\frac{\partial p_3}{\partial p_1} \quad \text{(D-30)}$$

$$y = -2z\frac{\partial p_3}{\partial p_2}, \quad \text{(D-31)}$$

where (x, y) denotes the source-receiver offset vector; z denotes the thickness of the horizontal layer; (p_1, p_2, p_3) denotes the phase slowness vector of incident P-waves; and the vertical slowness component p_3 as a function of the horizontal slowness components p_1 and p_2 is shown in equation B-13 with equations B-14 to B-16 of Appendix B. From equations D-30 and D-31, we find the radial source-receiver offset h_r given by

$$h_r = 2z\sqrt{\left(\frac{\partial p_3}{\partial p_1}\right)^2 + \left(\frac{\partial p_3}{\partial p_2}\right)^2}. \quad \text{(D-32)}$$

The slope $p_r = \partial t_r/\partial h_r$ of the traveltime curve t_r-h_r for the acquisition azimuth Φ is equivalent to the projection of the slowness vector on the acquisition direction:

$$p_r = p_1 \cos\Phi + p_2 \sin\Phi, \quad \text{(D-33)}$$

where the sine and cosine of the acquisition azimuth Φ are found from equations D-30 to D-32 as

$$\cos\Phi = -\frac{\partial p_3}{\partial p_1}\Big/\sqrt{\left(\frac{\partial p_3}{\partial p_1}\right)^2 + \left(\frac{\partial p_3}{\partial p_2}\right)^2} \quad \text{(D-34)}$$

$$\sin\Phi = -\frac{\partial p_3}{\partial p_2}\Big/\sqrt{\left(\frac{\partial p_3}{\partial p_1}\right)^2 + \left(\frac{\partial p_3}{\partial p_2}\right)^2}. \quad \text{(D-35)}$$

From these operations, it follows that all quantities in the right side of equation D-25 can be represented in terms of the horizontal slowness components p_1 and p_2. As the polar angle θ of the phase propagation direction from equation D-25 approaches $\pi/2$, the incident and reflected rays will become the horizontal ray. This means that Ω defined in equation D-25 is solely a function of the azimuth φ of the phase velocity in the $[x, y]$ plane of an orthorhombic medium. The approximate relationship between the azimuth φ of phase propagation direction and the azimuth Φ of group propagation direction can be obtained for P-waves defined in the $[x, y]$ plane of an orthorhombic medium (Appendix E). Consequently, for acoustic orthorhombic media, we derive the expression for $\Omega_{\text{acoustic}}(\Phi)$,

$$\Omega_{\text{acoustic}}(\Phi) = \frac{K}{\sqrt{LQ^5}}\frac{\Lambda}{(\sqrt{H}MQ\nu_{\text{P0}}^2 q + \sqrt{L}K)}, \quad \text{(D-36)}$$

where $\Omega_{\text{acoustic}}(\Phi)$ denotes the acoustic version of Ω defined in equation D-25; the quantity $q = q(\Phi)$ denotes the phase slowness squared of P-waves defined in the $[x, y]$ plane in terms of the azimuth Φ of group propagation direction, and the analytic expression for $q = q(\Phi)$ is given by equation E-6 of Appendix E; all quantities except ν_{P0} on the right side of equation D-36 are functions of the phase slowness squared q and the phase azimuth φ, and the analytic representation of $\varphi = \varphi(\Phi)$ is given by equation E-7 of Appendix E; these quantities are shown in the modified Alkhalifah's notation (Appendix B) as follows:

The function $K = K(q, \varphi)$:

$$K = \xi_3^6 + K_1\xi_3^6(\nu_{\text{P0}}^2 q) + K_2\xi_3^4(\nu_{\text{P0}}^2 q)^2 + K_3\xi_3^4(\nu_{\text{P0}}^2 q)^3 + K_4\xi_3^2(\nu_{\text{P0}}^2 q)^4 + K_5\xi_3^2(\nu_{\text{P0}}^2 q)^5 + K_6(\nu_{\text{P0}}^2 q)^6, \quad \text{(D-37)}$$

with

$$K_1 = 3(r_2(1-\xi_2^2)\cos^2\varphi + r_1(1-\xi_1^2)\sin^2\varphi), \quad \text{(D-38)}$$

$$K_2 = r_2^2\xi_3^2(2 - 5\xi_2^2 + 3\xi_2^4)\cos^4\varphi + r_1^2\xi_3^2(2 - 5\xi_1^2 + 3\xi_1^4)\sin^4\varphi + r_1 r_2(8\xi_1\xi_2\xi_3 - 3\xi_1^2\xi_2^2 + \xi_3^2(4 - 9\xi_1^2 - 9\xi_2^2 + 9\xi_1^2\xi_2^2))\cos^2\varphi\sin^2\varphi, \quad \text{(D-39)}$$

$$K_3 = -r_2^3\xi_2^2(1-\xi_2^2)^2\xi_3^2\cos^6\varphi - r_1^3\xi_1^2(1-\xi_1^2)^2\xi_3^2\sin^6\varphi + r_1 r_2^2(2\xi_1\xi_2(5-7\xi_2^2)\xi_3 - \xi_2^2(7-9\xi_2^2)\xi_3^2 - \xi_1^2(2-3\xi_2^2)(2\xi_2^2 + 3(1-\xi_2^2)\xi_3^2))\cos^4\varphi\sin^2\varphi - r_1^2 r_2(-2\xi_1\xi_2(5-7\xi_1^2)\xi_3 + 2\xi_1^2(2-3\xi_1^2)\xi_2^2 + (6\xi_2^2 + \xi_1^2(7-15\xi_2^2) - 9\xi_1^4(1-\xi_2^2))\xi_3^2)\cos^2\varphi\sin^4\varphi, \quad \text{(D-40)}$$

$$K_4 = 3r_1 r_2(2\xi_1\xi_2\xi_3 - \xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2)\xi_3^2)\cos^2\varphi\sin^2\varphi \times (r_2^2\xi_2^2\xi_3^2(\xi_2^2 - 1)\cos^4\varphi + r_1^2\xi_1^2\xi_3^2(\xi_1^2 - 1)\sin^4\varphi + r_1 r_2(2\xi_1\xi_2\xi_3 - \xi_1^2\xi_2^2 - 2(\xi_1^2 + \xi_2^2)\xi_3^2 + 3\xi_1^2\xi_2^2\xi_3^2)\cos^2\varphi\sin^2\varphi), \quad \text{(D-41)}$$

$$K_5 = -r_1^2 r_2^2 (2\xi_1\xi_2\xi_3 - \xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2)\xi_3^2) \times (r_2\xi_2^2(4\xi_1\xi_2\xi_3 - 2\xi_2^2\xi_3^2 \times + \xi_1^2(1 - 3\xi_3^2 + 3\xi_2^2(\xi_3^2 - 1)))\cos^2\varphi + r_1\xi_1^2(4\xi_1\xi_2\xi_3 - 2\xi_1^2\xi_3^2 + \xi_2^2(1 - 3\xi_3^2 + 3\xi_1^2(\xi_3^2 - 1)))\sin^2\varphi)\cos^4\varphi\sin^4\varphi, \quad \text{(D-42)}$$

$$K_6 = r_1^3 r_2^3 \xi_1^2\xi_2^2(-1 + \xi_3^2)(2\xi_1\xi_2\xi_3 - \xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2)\xi_3^2)^2\cos^6\varphi\sin^6\varphi. \quad \text{(D-43)}$$

The function $L = L(q, \varphi)$:

$$L = r_1^2(\xi_3 + r_2\nu_{P0}^2 q\xi_2(\xi_1 - \xi_2\xi_3)\cos^2\varphi)^4\sin^2\theta + r_2^2(\xi_3 + r_1\nu_{P0}^2 q\xi_1(\xi_2 - \xi_1\xi_3)\sin^2\varphi)^4\cos^2\theta. \quad \text{(D-44)}$$

The function $Q = Q(q, \varphi)$:

$$Q = \xi_3^2 + \nu_{P0}^2 q\xi_3^2(r_2(1 - \xi_2^2)\cos^2\varphi + r_1(1 - \xi_1^2)\sin^2\varphi) + r_1 r_2(\nu_{P0}^2 q)^2(2\xi_1\xi_2\xi_3 - \xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2)\xi_3^2)\cos^2\varphi\sin^2\varphi. \quad \text{(D-45)}$$

The function $H = H(q, \varphi)$:

$$H = \xi_3^4(H_0 + H_1(\nu_{P0}^2 q) + H_2(\nu_{P0}^2 q)^2 + H_3(\nu_{P0}^2 q)^3 + H_4(\nu_{P0}^2 q)^4), \quad \text{(D-46)}$$

with

$$H_0 = \xi_3^2(r_2^2\cos^2\varphi + r_1^2\sin^2\varphi), \quad \text{(D-47)}$$

$$H_1 = 4r_1 r_2\xi_3(r_2\xi_1(\xi_2 - \xi_1\xi_3) + r_1\xi_2(\xi_1 - \xi_2\xi_3))\cos^2\varphi\sin^2\varphi, \quad \text{(D-48)}$$

$$H_2 = 6r_1^2 r_2^2(\xi_2^2(\xi_1 - \xi_2\xi_3)^2\cos^2\varphi + \xi_1^2(\xi_2 - \xi_1\xi_3)^2\sin^2\varphi)\cos^2\varphi\sin^2\varphi, \quad \text{(D-49)}$$

$$H_3 = 4r_1^2 r_2^2\xi_3(r_2\xi_2^3(\xi_1 - \xi_2\xi_3)^3\cos^4\varphi + r_1\xi_1^3(\xi_2 - \xi_1\xi_3)^3\sin^4\varphi)\cos^2\varphi\sin^2\varphi, \quad \text{(D-50)}$$

$$H_4 = r_1^2 r_2^2(r_2^2\xi_2^4(\xi_1 - \xi_2\xi_3)^4\cos^6\varphi + r_1^2\xi_1^4(\xi_2 - \xi_1\xi_3)^4\sin^6\varphi)\cos^2\varphi\sin^2\varphi. \quad \text{(D-51)}$$

The function $M = M(q, \varphi)$:

$$M = \xi_3^2(r_2\cos^2\varphi + r_1\sin^2\varphi) + 2r_1 r_2\nu_{P0}^2 q\xi_3(2\xi_1\xi_2 - (\xi_1^2 + \xi_2^2)\xi_3)\cos^2\varphi\sin^2\varphi + r_1 r_2(\nu_{P0}^2 q)^2(r_2\xi_2^2(\xi_1 - \xi_2\xi_3)^2\cos^2\varphi + r_1\xi_1^2(\xi_2 - \xi_1\xi_3)^2\sin^2\varphi)\cos^2\varphi\sin^2\varphi. \quad \text{(D-52)}$$

The function $\Lambda = \Lambda(q, \varphi)$:

$$\Lambda = 2(\nu_{P0}^2 q)^4\xi_3^3(\xi_3(r_1\sin^2\varphi + r_2\cos^2\varphi) + r_1 r_2\nu_{P0}^2 q(2\xi_1\xi_2 - (\xi_1^2 + \xi_2^2)\xi_3)\sin^2\varphi\cos^2\varphi)^3 \times (\Lambda_0\xi_3^2 + \Lambda_1 r_1 r_2\xi_3^2\nu_{P0}^2 q + \Lambda_2 r_1 r_2(\nu_{P0}^2 q)^2 + \Lambda_3 r_1 r_2(\nu_{P0}^2 q)^3 + \Lambda_4 r_1 r_2(\nu_{P0}^2 q)^4 + \Lambda_5 r_1^2 r_2^2(\nu_{P0}^2 q)^5 + \Lambda_6 r_1^3 r_2^3(\nu_{P0}^2 q)^6), \quad \text{(D-53)}$$

with

$$\Lambda_0 = -r_2^2(15r_1 - r_2\xi_3^6)\cos^4\varphi - r_1^2(15r_2 - r_1\xi_3^6)\sin^4\varphi, \quad \text{(D-54)}$$

$$\Lambda_1 = -r_2^2\xi_1^2\cos^6\varphi - r_1^2\xi_2^2\sin^6\varphi + r_2\xi_3^4(15r_1\xi_2^2 + r_2\xi_1(\xi_1 + 6\xi_2\xi_3 - 6\xi_1\xi_3^2))\cos^4\varphi\sin^2\varphi + r_1\xi_3^4(15r_2\xi_1^2 + r_1\xi_2(\xi_2 + 6\xi_1\xi_3 - 6\xi_2\xi_3^2))\cos^2\varphi\sin^4\varphi, \quad \text{(D-55)}$$

$$\Lambda_2 = 15r_2^3\xi_2^4\xi_3^4\cos^8\theta + 15r_1^3\xi_1^4\xi_3^4\sin^8\varphi + r_1 r_2^2(15\xi_1^2\xi_2^2 + \xi_1^4\xi_3^4 + 20\xi_1\xi_2^3\xi_3^5 - 60\xi_2^4\xi_3^6)\cos^6\varphi\sin^2\varphi + r_1^2 r_2(15\xi_1^2\xi_2^2 + \xi_2^4\xi_3^4 + 20\xi_2\xi_1^3\xi_3^5 - 60\xi_1^4\xi_3^6)\cos^2\varphi\sin^6\varphi - r_1 r_2\xi_3^4(r_2\xi_1^3(\xi_1 - 6\xi_2\xi_3 + 4\xi_1\xi_3^2 + 30\xi_2\xi_3^3 - 15\xi_1\xi_3^4) + r_1\xi_2^3(\xi_2 - 6\xi_1\xi_3 + 4\xi_2\xi_3^2 + 30\xi_1\xi_3^3 - 15\xi_2\xi_3^4))\cos^4\varphi\sin^4\varphi, \quad \text{(D-56)}$$

$$\Lambda_3 = 3r_2^4\xi_2^5\xi_3^2(2\xi_1\xi_3 - 5\xi_2(1 + 2\xi_3^2))\cos^{10}\theta + 3r_1^4\xi_1^5\xi_3^2(2\xi_2\xi_3 - 5\xi_1(1 + 2\xi_3^2))\sin^{10}\theta + r_1 r_2^3\xi_2^2(\xi_1^4 + 15\xi_1^2\xi_2^2 - 60\xi_1\xi_2^3\xi_3^5 - 15\xi_2^4\xi_3^2(1 - \xi_3^2 - 6\xi_3^4))\cos^8\theta\sin^2\theta + r_1^3 r_2\xi_1^2(\xi_2^4 + 15\xi_1^2\xi_2^2 - 60\xi_1^3\xi_2\xi_3^5 - 15\xi_1^4\xi_3^2(1 - \xi_3^2 - 6\xi_3^4))\cos^2\theta\sin^8\theta + r_1 r_2^2(15r_2\xi_2^6\xi_3^2 + r_1(15\xi_1^4\xi_2^2 + 6\xi_1^5\xi_2\xi_3^3 - \xi_1^6\xi_3^2(1 + 2\xi_3^2) - 6\xi_1\xi_2^5\xi_3^3(1 + 3\xi_3^2 - 10\xi_3^4) + \xi_2^6\xi_3^2(1 + 3\xi_3^2 + 6\xi_3^4 - 20\xi_3^6)))\cos^6\theta\sin^4\theta + r_1^2 r_2(15r_1\xi_1^6\xi_3^2 + r_2(15\xi_1^2\xi_2^4 + 6\xi_1\xi_2^5\xi_3^3 - \xi_2^6\xi_3^2(1 + 2\xi_3^2) - 6\xi_1^5\xi_2\xi_3^3(1 + 3\xi_3^2 - 10\xi_3^4) + \xi_1^6\xi_3^2(1 + 3\xi_3^2 + 6\xi_3^4 - 20\xi_3^6)))\cos^4\theta\sin^6\theta, \quad \text{(D-57)}$$

$$\begin{aligned}\Lambda_4 = {} & r_2^5\xi_2^6\xi_3^2(\xi_1^2 - 6\xi_1\xi_2\xi_3 + 15\xi_2^2(2+\xi_3^2))\cos^{12}\theta \\ & + r_1^5\xi_1^6\xi_3^2(\xi_2^2 - 6\xi_1\xi_2\xi_3 + 15\xi_1^2(2+\xi_3^2))\sin^{12}\theta \\ & + r_1 r_2^4\xi_2^4(\xi_1^4 + 15\xi_1^2\xi_2^2 - 30\xi_2^4\xi_3^4(1+2\xi_3^2) \\ & + 20\xi_1\xi_2^3\xi_3^3(1+3\xi_3^2))\cos^{10}\theta\sin^2\theta \\ & + r_2 r_1^4\xi_1^4(\xi_2^4 + 15\xi_1^2\xi_2^2 - 30\xi_1^4\xi_3^4(1+2\xi_3^2) \\ & + 20\xi_2\xi_1^3\xi_3^3(1+3\xi_3^2))\cos^2\theta\sin^{10}\theta \\ & + r_1^2 r_2^3\xi_2^2(\xi_1^6 + 15\xi_1^4\xi_2^2 + 6\xi_1\xi_2^5\xi_3^3(2+3\xi_3^2-10\xi_3^4) \\ & - \xi_2^6\xi_3^2(2+3\xi_3^2+4\xi_3^4-15\xi_3^6))\cos^8\theta\sin^4\theta \\ & + r_1^3 r_2^2\xi_1^2(\xi_2^6 + 15\xi_1^2\xi_2^4 + 6\xi_1^5\xi_2\xi_3^3(2+3\xi_3^2-10\xi_3^4) \\ & - \xi_1^6\xi_3^2(2+3\xi_3^2+4\xi_3^4-15\xi_3^6))\cos^4\theta\sin^8\theta \\ & + r_1^2 r_2^2(r_1\xi_1^6(15\xi_2^2 - 6\xi_1\xi_2\xi_3^3 + \xi_1^2\xi_3^2(2+\xi_3^2)) \\ & + r_2\xi_2^6(15\xi_1^2 - 6\xi_1\xi_2\xi_3^3 + \xi_2^2\xi_3^2(2+\xi_3^2)))\cos^6\theta\sin^6\theta,\end{aligned} \tag{D-58}$$

$$\begin{aligned}\Lambda_5 = {} & r_2^4\xi_2^6(\xi_1^4 + 30\xi_1^2\xi_2^2 - 6\xi_1^3\xi_2\xi_3 - 20\xi_1\xi_2^3\xi_3^3(1+\xi_3^2) \\ & + 15\xi_2^4\xi_3^2(1+\xi_3^2+\xi_3^4))\cos^{10}\theta \\ & + r_1^4\xi_1^6(30\xi_1^2\xi_2^2 + \xi_2^4 - 6\xi_1\xi_2^3\xi_3 - 20\xi_1^3\xi_2\xi_3^3(1+\xi_3^2) \\ & + 15\xi_1^4\xi_3^2(1+\xi_3^2+\xi_3^4))\sin^{10}\theta \\ & + r_1 r_2^3\xi_2^4(\xi_1^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 - 6\xi_1\xi_2^5\xi_3^3(1+\xi_3^2-5\xi_3^4) \\ & + \xi_2^6\xi_3^2(1+\xi_3^2+\xi_3^4-6\xi_3^6))\cos^8\theta\sin^2\theta \\ & + r_1^3 r_2\xi_1^4(\xi_2^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 - 6\xi_1^5\xi_2\xi_3^3(1+\xi_3^2-5\xi_3^4) \\ & + \xi_1^6\xi_3^2(1+\xi_3^2+\xi_3^4-6\xi_3^6))\cos^2\theta\sin^8\theta \\ & + r_1^2 r_2^2\xi_1^2\xi_2^2(2\xi_1^6 + 15\xi_1^4\xi_2^2 - \xi_2^6 - 6\xi_1^5\xi_2\xi_3 \\ & + 6\xi_1\xi_2^5\xi_3)\cos^6\theta\sin^4\theta + r_1^2 r_2^2\xi_1^2\xi_2^2(2\xi_2^6 + 15\xi_1^2\xi_2^4 \\ & - \xi_1^6 - 6\xi_1\xi_2^5\xi_3 + 6\xi_1^5\xi_2\xi_3)\cos^4\theta\sin^6\theta,\end{aligned} \tag{D-59}$$

$$\begin{aligned}\Lambda_6 = {} & r_2^3\xi_2^6(\xi_1^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 - 6\xi_1^5\xi_2\xi_3 - 20\xi_1^3\xi_2^3\xi_3 \\ & - 6\xi_1\xi_2^5\xi_3^7 + \xi_2^6\xi_3^8)\cos^8\theta + r_1^3\xi_1^6(\xi_2^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 \\ & - 6\xi_1\xi_2^5\xi_3 - 20\xi_1^3\xi_2^3\xi_3 - 6\xi_1^5\xi_2\xi_3^7 + \xi_1^6\xi_3^8)\sin^8\theta \\ & + r_1 r_2^2\xi_1^2\xi_2^4(\xi_1^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 + \xi_2^6 \\ & - 2\xi_1\xi_2(3\xi_1^2+\xi_2^2)(\xi_1^2+3\xi_2^2)\xi_3)\cos^6\theta\sin^2\theta \\ & + r_1^2 r_2\xi_1^4\xi_2^2(\xi_1^6 + 15\xi_1^4\xi_2^2 + 15\xi_1^2\xi_2^4 + \xi_2^6 \\ & - 2\xi_1\xi_2(3\xi_1^2+\xi_2^2)(\xi_1^2+3\xi_2^2)\xi_3)\cos^2\theta\sin^6\theta.\end{aligned} \tag{D-60}$$

Next, we derive the expression for Ω defined in equation D-26 for elastic orthorhombic media by the perturbation method. An elastic orthorhombic medium is taken into account as the perturbation from the reference of the corresponding acoustic orthorhombic medium. The S-wave velocity parameter ν_{S0} defined in equation A-2 of Appendix A is taken as the perturbation parameter. For acoustic orthorhombic media, Ω defined in equation D-26 is reduced to Ω_{acoustic}, which is given by equation D-36 with equations D-37 to D-60. For the $[x, z]$ and $[y, z]$ planes of an orthorhombic medium, we derive the perturbation term Ω_{elastic} corresponding to the change in density-normalized stiffness coefficients from acoustic to elastic orthorhombic media,

$$\Omega_{\text{elastic}}(\Phi=0) = \frac{c_{55}(c_{11}-c_{33})(c_{33}-c_{55})(c_{13}^2-c_{11}c_{33}+(c_{11}+2c_{13}+c_{33})c_{55})}{c_{33}(c_{13}^2+2c_{13}c_{55}+c_{33}c_{55})^2} \tag{D-61}$$

$$\Omega_{\text{elastic}}(\Phi=\pi/2) = \frac{c_{44}(c_{22}-c_{33})(c_{33}-c_{44})(c_{23}^2-c_{22}c_{33}+(c_{22}+2c_{23}+c_{33})c_{44})}{c_{33}(c_{23}^2+2c_{23}c_{44}+c_{33}c_{44})^2}. \tag{D-62}$$

The term $\Omega_{\text{elastic}}(\Phi)$ in equation D-26 is assumed to satisfy an elliptic function defined as

$$\Omega_{\text{elastic}}(\Phi) = \Omega_{\text{elastic}}(\Phi=0)\cos^2\Phi + \Omega_{\text{elastic}}(\Phi=\pi/2)\sin^2\Phi. \tag{D-63}$$

APPENDIX E

THE ANALYTIC APPROXIMATIONS FOR THE PHASE SLOWNESS AND ITS PHASE ANGLE OF P-WAVES IN THE HORIZONTAL PLANE OF AN ORTHORHOMBIC MEDIUM

The goal of this appendix is to analytically express the squared magnitude of phase slowness q and the azimuth of the phase propagation direction φ in terms of the azimuth of group propagation direction Φ for P-waves in the horizontal plane of an orthorhombic medium.

For P-waves in the horizontal ($[x, y]$) plane of an orthorhombic medium, the slowness vector $(p_1, p_2, 0)$ is defined as

$$p_1 = \frac{\partial T}{\partial x}, \qquad p_2 = \frac{\partial T}{\partial y}. \tag{E-1}$$

Here, T denotes the traveltime from the coordinate origin $(0,0,0)$ to the position $(x, y, 0)$,

$$T = \frac{\sqrt{x^2+y^2}}{\nu_h(\Phi)}, \quad \Phi = \arctan(y/x), \tag{E-2}$$

where the group velocity $\nu_h(\Phi)$ is defined in the $[x, y]$ plane, and its analytic approximation is given by

$$\nu_h(\Phi) = 1/\sqrt{N_0(\Phi)}, \tag{E-3}$$

where $N_0(\Phi)$ denotes the approximate group slowness squared in the $[x, y]$ plane, given by equations D-9 to D-13 of Appendix D for elastic orthorhombic media and equation D-27 of Appendix D for acoustic orthorhombic media.

Substitution of the traveltime formula E-2 into the definition of slowness vector E-1 yields

$$p_1(\Phi) = \frac{\cos\Phi}{\nu_h(\Phi)} - \frac{1}{2}\nu_h(\Phi)\Upsilon(\Phi)\sin\Phi,$$
$$p_2(\Phi) = \frac{\sin\Phi}{\nu_h(\Phi)} + \frac{1}{2}\nu_h(\Phi)\Upsilon(\Phi)\cos\Phi, \tag{E-4}$$

with

$$\Upsilon(\Phi) = \frac{\partial(1/\nu_h^2)}{\partial\Phi} = \frac{(c_{11}-a_{SF})(1-\zeta)\sin(2\Phi)}{c_{11}a_{SF}} - \frac{\zeta(c_{SF}^2(b_{SF}-c_{11})\cos^2\Phi + c_{11}(c_{SF}^2 - c_{11}b_{SF})\sin^2\Phi)\sin(2\Phi)}{c_{11}^2 b_{SF} c_{SF}^2\sqrt{\frac{\cos^4\Phi}{c_{11}^2}+\frac{2\cos^2\Phi\sin^2\Phi}{c_{11}b_{SF}}+\frac{\sin^4\Phi}{c_{SF}^2}}}, \tag{E-5}$$

where ζ, a_{SF}, b_{SF}, and c_{SF} are given by equations D-10 to D-13 of Appendix D for elastic orthorhombic media and equations D-15 to D-17 of Appendix D for acoustic orthorhombic media; c_{11} is expressed by equation D-14 of Appendix D for acoustic orthorhombic media.

The squared magnitude of the slowness vector $q = p_1^2 + p_2^2$ can thus be written as

$$q(\Phi) = \frac{1}{\nu_h^2(\Phi)} + \frac{1}{4}\nu_h^2(\Phi)\Upsilon^2(\Phi), \tag{E-6}$$

corresponding to the phase azimuth φ as a function of the group azimuth Φ, given as

$$\varphi = \Phi + \arctan\left(\frac{\nu_h^2(\Phi)\Upsilon(\Phi)}{2}\right). \tag{E-7}$$

REFERENCES

Al-Dajani, A., and M. N. Toksoz, 2002, Nonhyperbolic reflection moveout for orthorhombic media: MIT Earth Resources Laboratory Industry Consortia annual report, http://hdl.handle.net/1721.1/67847, accessed 21 December 2011.

Alkhalifah, T., 1998, Acoustic approximations for seismic processing in transversely isotropic media: Geophysics, **63**, 623–631, doi: 10.1190/1.1444361.

Alkhalifah, T., 2000, The offset-midpoint traveltime pyramid in transversely isotropic media: Geophysics, **65**, 1316–1325, doi: 10.1190/1.1444823.

Alkhalifah, T., 2003, An acoustic wave equation for orthorhombic anisotropy: Geophysics, **68**, 1169–1172, doi: 10.1190/1.1598109.

Backus, G. E., 1965, Possible forms of seismic anisotropy of the uppermost mantle under oceans: Journal of Geophysical Research, **70**, 3429–3439, doi: 10.1029/JZ070i014p03429.

Berryman, J. G., 2008, Exact seismic velocities for transversely isotropic media and extended Thomsen formulas for stronger anisotropies: Geophysics, **73**, no. 1, D1–D10, doi: 10.1190/1.2813433.

Červený, V., 2001, Seismic ray theory: Cambridge University Press.

Daley, P. F., and E. S. Krebes, 2004a, Alternative linearized expressions for qP, qS_1 and qS_2 phase velocities in a weakly anisotropic orthorhombic medium: Consortium for Research in Elastic Wave Exploration Seismology Research Report 16, 1–19.

Daley, P. F., and E. S. Krebes, 2004b, Approximate QL phase and group velocities in weakly orthorhombic anisotropic media: Consortium for Research in Elastic Wave Exploration Seismology Research Report 16, 1–18.

Daley, P. F., E. S. Krebes, and L. R. Lines, 2004, Phase velocity approximations in a transversely isotropic medium: CREWES Research Report 16, 1–16.

Dellinger, J., F. Muir, and M. Karrenbach, 1993, Anelliptic approximations for TI media: Journal of Seismic Exploration, **2**, 23–40.

Eaton, D. W. S., 1993, Finite difference traveltime calculation for anisotropic media: Geophysical Journal International, **114**, 273–280, doi: 10.1111/j.1365-246X.1993.tb03915.x.

Every, A. G., 1980, General closed-form expressions for acoustic waves in elastically anisotropic solids: Physical Review B, **22**, 1746–1760, doi: 10.1103/PhysRevB.22.1746.

Farra, V., 2001, High-order perturbations of the phase velocity and polarization of qP and qS waves in anisotropic media: Geophysical Journal International, **147**, 93–104, doi: 10.1046/j.1365-246X.2001.00510.x.

Farra, V., 2004, Improved first-order approximation of group velocities in weakly anisotropic media: Studia Geophysica et Geodaetica, **48**, 199–213, doi: 10.1023/B:SGEG.0000015592.36894.3b.

Farra, V., and I. Pšenčík, 2003, Properties of the zero-, first- and higher-order approximations of attributes of elastic waves in weakly anisotropic media: Journal of the Acoustical Society of America, **114**, 1366–1378, doi: 10.1121/1.1591772.

Farra, V., and I. Pšenčík, 2013, Moveout approximations for P and SV waves in VTI media: Geophysics, **78**, no. 5, WC81–WC92, doi: 10.1190/geo2012-0408.1.

Fomel, S., 2004, On anelliptic approximations for qP velocities in VTI media: Geophysical Prospecting, **52**, 247–259, doi: 10.1111/j.1365-2478.2004.00413.x.

Fomel, S., and A. Stovas, 2010, Generalized nonhyperbolic moveout approximation: Geophysics, **75**, no. 2, U9–U18, doi: 10.1190/1.3334323.

Golikov, P., and A. Stovas, 2012, Accuracy comparison of nonhyperbolic moveout approximations for qP-waves in VTI media: Journal of Geophysics and Engineering, **9**, 428–432, doi: 10.1088/1742-2132/9/4/428.

Grechka, V., and I. Tsvankin, 1999, 3-D moveout velocity analysis and parameter estimation for orthorhombic media: Geophysics, **64**, 820–837, doi: 10.1190/1.1444593.

Han, W., and S. Xu, 2012, Raytracing and traveltime calculations for orthorhombic anisotropic media: 74th Annual International Conference and Exhibition, EAGE, Extended Abstracts, A023.

Jech, J., and I. Pšenčík, 1989, First-order perturbation method for anisotropic media: Geophysical Journal International, **99**, 369–376, doi: 10.1111/j.1365-246X.1989.tb01694.x.

Mah, M., and D. R. Schmitt, 2003, Determination of the complete elastic stiffnesses from ultrasonic phase velocity measurements: Journal of Geophysical Research, **108**, 2016–2027, doi: 10.1029/2001JB001586.

Mahmoudian, F., G. F. Margrave, P. F. Daley, J. Wong, and D. C. Henley, 2014, Estimation of elastic stiffness coefficients of an orthorhombic physical model using group velocity analysis on transmission data: Geophysics, **79**, no. 1, R27–R39, doi: 10.1190/geo2013-0203.1.

McMechan, G. A., 1983, Seismic tomography in boreholes: Geophysical Journal International, **74**, 601–612, doi: 10.1111/j.1365-246X.1983.tb01891.x.

Mensch, T., and V. Farra, 1999, Computation of qP-wave rays, traveltimes and slowness vectors in orthorhombic media: Geophysical Journal International, **138**, 244–256, doi: 10.1046/j.1365-246x.1999.00870.x.

Mensch, T., and P. Rasolofosaon, 1997, Elastic-wave velocities in anisotropic media of arbitrary anisotropy — Generalization of Thomsen's parameters ε, δ and γ: Geophysical Journal International, **128**, 43–64, doi: 10.1111/j.1365-246X.1997.tb04070.x.

Miller, D. E., and C. Spencer, 1994, An exact inversion for anisotropic moduli from phase slowness data: Journal of Geophysical Research, **99**, 651–657.

Pšenčík, I., and D. Gajewski, 1998, Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media: Geophysics, **63**, 1754–1766, doi: 10.1190/1.1444470.

Pšenčík, I., and V. Vavrycuk, 2002, Approximate relation between the ray vector and wave normal directions in weakly anisotropic media: Studia Geophysica et Geodaetica, **46**, 793–807, doi: 10.1023/A:1021189724526.

Sano, O., Y. Kudo, and Y. Mizuta, 1992, Experimental determination of elastic constants of Oshima granite, Barre granite, and Chelmsford granite: Journal of Geophysical Research, **97**, 3367–3379, doi: 10.1029/91JB02934.

Schoenberg, M., and K. Helbig, 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954–1974, doi: 10.1190/1.1444297.

Song, L.-P., and A. G. Every, 2000, Approximate formulae for acoustic wave group slowness in weakly orthorhombic media: Journal of Physics D: Applied Physics, **33**, L81–L85, doi: 10.1088/0022-3727/33/17/101.

Song, L.-P., A. G. Every, and C. Wright, 2001, Linearized approximations for phase velocities of elastic waves in weakly anisotropic media: Journal of Physics D: Applied Physics, **34**, 2052–2062, doi: 10.1088/0022-3727/34/13/316.

Song, X., and T. Alkhalifah, 2013, Modeling of pseudoacoustic P-waves in orthorhombic media with a low-rank approximation: Geophysics, **78**, no. 4, C33–C40, doi: 10.1190/geo2012-0144.1.

Sripanich, Y., and S. Fomel, 2014, Anelliptic approximations for qP velocities in TI and orthorhombic media: 84th Annual International Meeting, SEG, Expanded Abstracts, 453–457.

Stovas, A., 2010, Generalized moveout approximation for qP- and qSV-waves in a homogeneous transversely isotropic medium: Geophysics, **75**, no. 6, D79–D84, doi: 10.1190/1.3506504.
Stovas, A., and S. Fomel, 2012, Generalized nonelliptic moveout approximation in τ-p domain: Geophysics, **77**, no. 2, U23–U30, doi: 10.1190/geo2011-0119.1.
Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966, doi: 10.1190/1.1442051.
Tsvankin, I., 1997, Anisotropic parameters and P-wave velocity for orthorhombic media: Geophysics, **62**, 1292–1309, doi: 10.1190/1.1444231.
Ursin, B., and A. Stovas, 2006, Traveltime approximations for a layered transversely isotropic medium: Geophysics, **71**, no. 2, D23–D33, doi: 10.1190/1.2187716.
Vasconcelos, I., and I. Tsvankin, 2006, Non-hyperbolic moveout inversion of wide-azimuth P-wave data for orthorhombic media: Geophysical Prospecting, **54**, 535–552, doi: 10.1111/j.1365-2478.2006.00559.x.
Williamson, P. R., 1990, Tomographic inversion in reflection seismology: Geophysical Journal International, **100**, 255–274, doi: 10.1111/j.1365-246X.1990.tb02484.x.
Xu, S., and H. Zhou, 2014, Accurate simulations of pure quasi-P-waves in complex anisotropic media: Geophysics, **79**, no. 6, T341–T348, doi: 10.1190/geo2014-0242.1.
Xu, X., I. Tsvankin, and A. Pech, 2005, Geometrical spreading of P-waves in horizontally layered, azimuthally anisotropic media: Geophysics, **70**, no. 5, D43–D53, doi: 10.1190/1.2052467.