GEOPHYSICS, VOL. 81, NO. 3 (MAY-JUNE 2016); P. C99–C112, 12 FIGS., 1 TABLE.
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P-wave slowness surface approximation for tilted orthorhombic media

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ABSTRACT

We have developed an analytic and approximate formula for vertical slowness components of down- and upgoing plane P waves in 3D tilted orthorhombic media. A perturbation method and Shanks transform were used to derive the approximation for slowness surface of P waves in tilted orthorhombic media. We have also quantitatively described the validity range of the radial horizontal slowness components for the proposed formula. The validity range was affected by the strength of the anellipticity of an orthorhombic medium: the stronger the anellipticity, the smaller the validity range. Numerical examples determined that the proposed formula is accurate for tilted orthorhombic media with weak to strong anellipticity. We have also evaluated in detail the application of the proposed formula on calculating the P-wave intercept time in the τ-p domain for horizontally layered, tilted orthorhombic models. Our formula is useful for ray tracing, phase-shift migration, and τ-p domain intercept time approximation for tilted orthorhombic media.

INTRODUCTION

Orthorhombic anisotropy is a common phenomenon of seismic wave propagation in sedimentary basins with parallel vertical fractures or isotropic media with two orthogonal sets of vertical fractures (Schoenberg and Helbig, 1997; Bakulin et al., 2000). An orthorhombic medium is characterized by three mutually orthogonal planes of mirror symmetry (Tsvankin, 1997). In each symmetry plane, the medium behaves in a transversely isotropic manner. Such media are generally described in Thomsen-type notation including nine independent parameters (Tsvankin, 1997). However, it is difficult to invert for all nine parameters from surface P-wave data provided only using traveltime. The main reason is that some parameters in Thomsen-type notation, related to S-waves, almost do not affect P-wave phase and group velocities. A common way of describing the P-wave slowness surface in orthorhombic media is using Alkhalifah's (2003) notation. In this notation, there remain six independent nonzero parameters and the S-wave velocities along the three symmetry axes are assumed to be zeros. Because the six parameters are linked to the parameters in Thomsen-type notation, the P-wave slowness surface under acoustic approximation can also be described by the six parameters in Thomsen-type notation without shear parameters γ_1 and γ_2. Besides, one can also use the weak anisotropy notation (e.g., Psencik and Farra, 2005) to describe the P-wave velocities and traveltimes for orthorhombic media.

Analytic representation of the vertical slowness component of plane P waves is useful for ray tracing, phase shift migration, and τ-p domain intercept time approximation for anisotropic media. For body-wave ray tracing in a multilayered anisotropic medium, Snell's law must be taken into account to calculate the slowness components of reflected and transmitted waves, normal to the local interface (e.g., Červený [2001], pp. 42–44; Vanelle and Gajewski, 2009). For prestack phase-shift migration, the phase-shift operator is connected with the vertical slowness components of downgoing and upgoing plane waves (e.g., Gazdag, 1978; Yilmaz [2001], pp. 498–500). For horizontally layered media, the intercept time in τ-p domain can be expressed by the summation over the product of layer thickness and absolute magnitudes of vertical slowness components of downgoing and upgoing plane waves (e.g., van der Baan and Kendall, 2002, 2003; Sen and Mukherjee, 2003; van der Baan, 2004; Sil and Sen, 2008, 2009).

For orthorhombic media with a vertical symmetry axis, it is not difficult to analytically calculate the vertical slowness component of plane P waves from the slowness surface equation (Alkhalifah, 2003). For general 3D tilted orthorhombic media, however, it is impossible to obtain the exact and analytic formula for the vertical slowness component of plane P waves. Perturbation method can help to obtain the analytic approximation for the vertical slowness component. Vanelle and Gajewski (2009) develop an approximation of the vertical slowness component for weakly anisotropic media. Stovas and Alkhalifah (2013) use a perturbation method and

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Shanks transform (Bender and Orszag [1978], pp. 369–375) to derive an analytic formula for the vertical slowness component of plane P waves in 2D tilted transversely isotropic media. For vertical transversely isotropic media, accurate rational approximations for vertical slowness component of plane P- and SV-waves are presented in Schoenberg and de Hoop (2000) and Pedersen et al. (2007).

The goal of this paper is to derive an analytic formula for the vertical slowness component of down-and upgoing plane P waves in tilted orthorhombic media. A titled orthorhombic medium in the frame of global Cartesian reference corresponds to the orthorhombic medium with three symmetry planes orthogonal to the local axes in the frame of local Cartesian reference. For simplicity, the two orthorhombic media are referred to as the tilted orthorhombic medium and the vertical orthorhombic medium throughout this paper. We account for the orthorhombic media under an acoustic assumption (Alkhalifah, 2003). To extend the approach proposed by Stovas and Alkhalifah (2013) to 3D orthorhombic media, we modify Alkhalifah's (2003) notation by replacing the parameter δ_3 in his notation by a new anellipticity parameter. Three anellipticity parameters defined in symmetry planes of an orthorhombic medium are involved in the notation after modification. This is very helpful for deriving our analytic formula, because the P-wave slowness surface in orthorhombic media is an ellipsoid when all three anellipticity parameters are zeros and the expression for the vertical slowness component is extremely straightforward in this case. We use a perturbation method to approximate the vertical slowness component of plane P waves. We take elliptically orthorhombic media as a background and three anellipticity parameters as perturbation parameters. The perturbation of the vertical slowness component with respect to three anellipticity parameters is determined from the slowness surface equation. Shanks transform (Bender and Orszag [1978], pp. 369–375) can accelerate the convergence of series like our perturbation expansion. Therefore, we apply the Shanks transform (Bender and Orszag [1978], pp. 369–375) to improve the accuracy of our perturbation expansion. The whole process of deriving vertical slowness component is straightforward. In the main text, the sign of vertical slowness component is assumed to be positive for downgoing plane P waves and negative for upgoing plane P waves.

PARAMETERIZATION FOR ORTHORHOMBIC MEDIA

We modify Alkhalifah's (2003) notation and define the following parameters to parameterize an acoustic orthorhombic medium with three symmetry planes orthogonal to the three axes of Cartesian coordinate system:

$$v_{p0} \equiv \sqrt{a_{33}}, \tag{1}$$

$$\eta_1 \equiv \frac{\varepsilon_1 - \delta_1}{1 + 2\delta_1} = \frac{a_{22}(a_{33} - a_{44})}{2(a_{23}^2 + a_{23}a_{44} + a_{33}a_{44})} - \frac{1}{2}, \tag{2}$$

$$\eta_2 \equiv \frac{\varepsilon_2 - \delta_2}{1 + 2\delta_2} = \frac{a_{11}(a_{33} - a_{55})}{2(a_{13}^2 + a_{13}a_{55} + a_{33}a_{55})} - \frac{1}{2}, \tag{3}$$

$$\eta_3 \equiv \frac{\varepsilon_1 - \varepsilon_2 - \delta_3(1+2\varepsilon_2)}{(1+2\varepsilon_2)(1+2\delta_3)} = \frac{a_{22}(a_{11} - a_{66})}{2(a_{12}^2 + 2a_{12}a_{66} + a_{11}a_{66})} - \frac{1}{2}, \tag{4}$$

$$r_1 \equiv 1 + 2\delta_1 = \frac{(a_{23} + a_{44})^2}{a_{33}(a_{33} - a_{44})} + \frac{a_{44}}{a_{33}}, \tag{5}$$

$$r_2 \equiv 1 + 2\delta_2 = \frac{(a_{13} + a_{55})^2}{a_{33}(a_{33} - a_{55})} + \frac{a_{55}}{a_{33}}, \tag{6}$$

where a_{ij} denote the density-normalized stiffness coefficients in Voigt notation; v_{p0} denotes the phase velocity of P waves along the vertical axis (z-axis); subscripts 1, 2, 3 except for a_{ij} correspond to the [y, z], [x, z], and [x, y] symmetry planes of an orthorhombic medium, respectively; ε_i $(i = 1,2)$ and δ_i $(i = 1,2,3)$ are Thomsen-type parameters for orthorhombic media (Tsvankin, 1997); η_i $(i = 1,2,3)$ denote the anellipticity parameters (Grechka and Tsvankin, 1999); and r_i $(i = 1,2)$ denote the factors for the NMO velocities squared.

EXACT SLOWNESS SURFACE

The slowness surface for tilted orthorhombic media can be obtained by rotating the slowness surface for vertical orthorhombic media. Let us define $\boldsymbol{p}_{\mathrm{v}} = (p_{\mathrm{v}1}, p_{\mathrm{v}2}, q_{\mathrm{v}})^T$ and $\boldsymbol{p} = (p_1, p_2, q)^T$ as the slowness vectors of P waves in vertical and tilted orthorhombic media, respectively. Considering the notation in the previous section, the P-wave slowness surface for acoustic orthorhombic media with the three symmetry planes orthogonal to the three axes of the local Cartesian coordinate system (Alkhalifah, 2003) is written as

$$F(p_{\mathrm{v}1}, p_{\mathrm{v}2}, q_{\mathrm{v}}) = v_{p0}^2 q_{\mathrm{v}}^2 f_2(p_{\mathrm{v}1}, p_{\mathrm{v}2}) - f_1(p_{\mathrm{v}1}, p_{\mathrm{v}2}) = 0, \tag{7}$$

where functions $f_1(p_{\mathrm{v}1}, p_{\mathrm{v}2})$ and $f_2(p_{\mathrm{v}1}, p_{\mathrm{v}2})$ are given by

$$f_1(p_{\mathrm{v}1}, p_{\mathrm{v}2}) = (1 - r_1\xi_1^2 p_{\mathrm{v}2}^2 v_{p0}^2)(1 - r_2\xi_2^2 p_{\mathrm{v}1}^2 v_{p0}^2) - \frac{1}{\xi_3^2} r_1 r_2 \xi_1^2 \xi_2^2 p_{\mathrm{v}1}^2 p_{\mathrm{v}2}^2 v_{p0}^4, \tag{8}$$

$$f_2(p_{\mathrm{v}1}, p_{\mathrm{v}2}) = 1 + r_1(1 - \xi_1^2) p_{\mathrm{v}2}^2 v_{p0}^2 + r_2(1 - \xi_2^2) p_{\mathrm{v}1}^2 v_{p0}^2 - r_1 r_2 \Omega p_{\mathrm{v}1}^2 p_{\mathrm{v}2}^2 v_{p0}^4, \tag{9}$$

with

$$\Omega = \xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2 + \frac{\xi_1^2\xi_2^2}{\xi_3^2} - \frac{2\xi_1\xi_2}{\xi_3}, \tag{10}$$

$$\xi_1 \equiv \sqrt{1 + 2\eta_1}, \quad \xi_2 \equiv \sqrt{1 + 2\eta_2}, \quad \xi_3 \equiv \sqrt{1 + 2\eta_3}. \tag{11}$$

The P-wave slowness surface for a tilted orthorhombic medium is linked to the corresponding slowness surface for the vertical orthorhombic medium through three successive rotations (e.g., Zhang and Zhang, 2011; Lapilli and Fowler, 2013). These rotations are expressed in matrix forms

$$\boldsymbol{R}_a = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_b = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$\boldsymbol{R}_c = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

where the definition of Euler angles ϕ, θ, and ψ is shown in Figure 1. Applying these rotations to the slowness surface for tilted orthorhombic media, one can obtain the slowness surface for the vertical orthorhombic media

$$\boldsymbol{p}_{\mathrm{v}} = \boldsymbol{R}_c\boldsymbol{R}_b\boldsymbol{R}_a\boldsymbol{p}, \tag{13}$$

where $\boldsymbol{p}_{\mathrm{v}} = (p_{\mathrm{v}1}, p_{\mathrm{v}2}, q_{\mathrm{v}})^T$ and $\boldsymbol{p} = (p_1, p_2, q)^T$ denote the slowness vectors in the orthorhombic medium with the three symmetry planes orthogonal to the three axes of the local Cartesian coordinate system and the tilted orthorhombic medium, respectively.

The radial horizontal slowness p_r and its azimuth φ measured from the x-axis are introduced to express the horizontal slowness components for tilted orthorhombic media

$$p_1 = p_r \cos\varphi, \qquad p_2 = p_r \sin\varphi. \tag{14}$$

Substituting equation 13 with equations 12 and 14 into equation 7 with equations 8–11, we can obtain a sixth-order polynomial equation in the vertical slowness component q, which is symbolically denoted by

$$G(q, p_r, \varphi) = 0. \tag{15}$$

The sixth-order polynomial equation generally has no exact and analytic solution. Assuming that the given p_r is in the range of the radial horizontal slowness components of P waves, we can always numerically obtain a pair of real roots from equation 15, of which the positive and negative roots correspond to the down- and upgoing P waves, respectively. However, we cannot guarantee that all other roots are always complex-valued. For acoustic transversely isotropic media with a vertical symmetry axis, for instance, the phase velocity of pseudo S-waves may be real-valued (Grechka et al., 2004). To identify the correct root of the considered P-wave, we can use the following procedure: (1) for a given horizontal slowness p_r, we calculate all six roots from equation 15, and we obtain the corresponding phase velocities; (2) we remain all real roots and neglect the complex-valued roots; (3) for each real root, we calculate the phase propagation direction; (4) we substitute the phase propagation direction to the exact formula for the P-wave phase velocity (e.g., Schoenberg and Helbig, 1997), to obtain the phase-velocity corresponding to this direction; and (5) if the phase velocities obtained in steps (1) and (4) are the same, we conclude that the root corresponds to a P-wave and determine the up- or down-propagation direction from the sign of this root; otherwise, we will neglect it and start with step (3) to check one other real root. It is worth noting that we can also use this procedure to identify the vertical slowness component of up- and downgoing P waves in elastic orthorhombic media with tilted symmetry axes.

VERTICAL SLOWNESS APPROXIMATION

In this section, we use a perturbation method and Shanks transform to find an approximate solution of equation 15. For a fixed pair

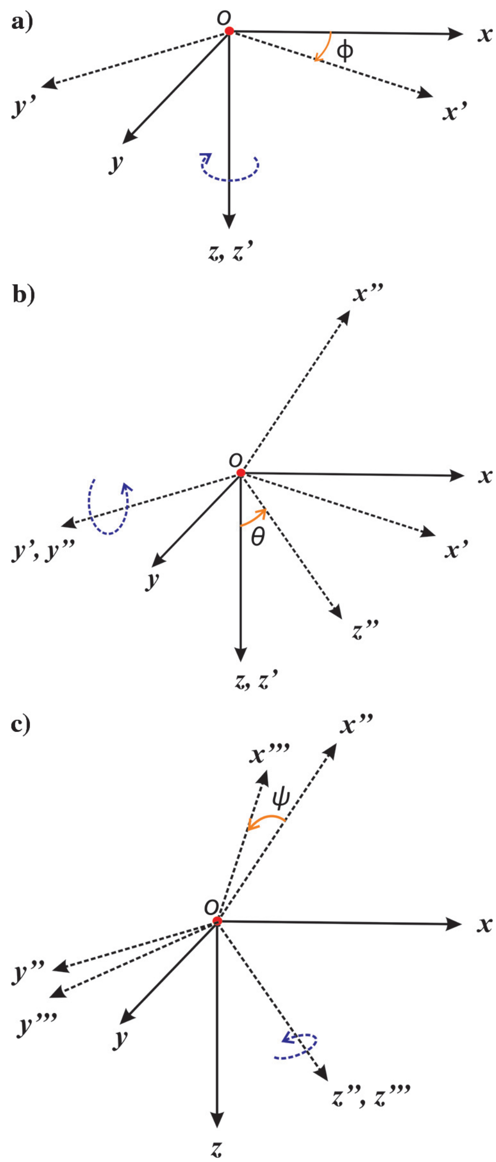

Figure 1. Schematic plots for the three independent coordinate rotations (after Lapilli and Fowler, 2013). The plots from left to right correspond to the rotations around the z-, y'-, and z''-axes, respectively. The (x, y, z) denotes the original Cartesian system corresponding to the slowness surface of P waves in tilted orthorhombic media. (x', y', z') and (x'', y'', z'') denote the Cartesian systems after first and second rotations around the z-axis and the y'-axis, respectively. (x''', y''', z''') denotes the final Cartesian system after the rotation around the z''-axis. The x'''-, y'''-, and z'''-axes are also three principle axes of orthorhombic media. In equation 13, slowness vectors $\boldsymbol{p}$ and $\boldsymbol{p}_{\mathrm{v}}$ are defined in the coordinate systems (x, y, z) and (x''', y''', z'''), respectively.

of the radial horizontal slowness p_r and its azimuth φ, we define the trial solution for the vertical slowness component with respect to the anellipticity parameters η_i, $i = 1,2,3$

$$q = q_0 + \sum_{i=1}^{3} q_i \eta_i + \sum_{i,j=1,i\le j}^{3} q_{ij}\eta_i\eta_j. \quad (16)$$

The anellipticity parameters η_i, $i = 1,2,3$, are involved in the slowness surface equation 15. The similar expansion of equation 15 is written as

$$G(q) = G_0(q) + \sum_{i=1}^{3} G_i(q)\eta_i + \sum_{i,j=1,i\le j}^{3} G_{ij}(q)\eta_i\eta_j = 0. \quad (17)$$

Here, the arguments p_r and φ in functions G, G_0, G_i, and G_{ij} are omitted for simplicity.

Equation 17 is a six-order polynomial equation in the unknown vertical slowness component q. By substituting the trial solution 16 into equation 17, we can obtain a new expansion with respect to the anellipticity parameters

$$G(q) = H_0(q_0) + \sum_{i=1}^{3} H_i(q_0, q_i)\eta_i + \sum_{i,j=1,i\le j}^{3} H_{ij}(q_0, q_1, q_2, q_3, q_{ij})\eta_i\eta_j = 0, \quad (18)$$

where H_i and H_{ij} are the polynomial functions.

From equation 18, we obtain

$$H_0(q_0) = 0, \quad H_i(q_0, q_i) = 0,$$
$$H_{ij}(q_0, q_1, q_2, q_3, q_{ij}) = 0, \quad i, j = 1,2,3, \text{and } i \le j. \quad (19)$$

Solving equation 19 successively, we determine all perturbation coefficients defined in the trial solution 16. Furthermore, we find that the trial solution 16 can be written as

$$q^{(\pm)}(p_r, \varphi) = \frac{1}{v_{p0}} \left(\tilde{q}_0^{(\pm)}(p_r, \varphi) + \sum_{i=1}^{3} \tilde{q}_i^{(\pm)}(p_r, \varphi)\eta_i + \sum_{i,j=1,i\le j}^{3} \tilde{q}_{ij}^{(\pm)}(p_r, \varphi)\eta_i\eta_j \right), \quad (20)$$

where $\tilde{q}_0^{(\pm)}$, $\tilde{q}_i^{(\pm)}$, and $\tilde{q}_{ij}^{(\pm)}$ are zero-, first-, and second-order dimensionless coefficients, which are the functions of p_r and φ; superscripts "+" and "−" correspond to the upper and lower parts of a slowness surface, that is, $q^{(+)} > q^{(-)}$ for each fixed pair of p_r and φ; the expressions for these coefficients are shown in Appendix A; for elliptically orthorhombic media ($\eta_1 = \eta_2 = \eta_3 = 0$), the first- and second-order coefficients $\tilde{q}_i^{(\pm)}$ and $\tilde{q}_{ij}^{(\pm)}$ in equation 20 vanish.

We apply the Shanks transform (Bender and Orszag [1978], pp. 369–375) to accelerate the convergence of expansion 20. Consequently, the final formula for the vertical slowness component becomes

$$q^{(\pm)}(p_r,\varphi) = \frac{1}{v_{p0}} \left(Q_0^{(\pm)}(p_r,\varphi) + \frac{\left(Q_1^{(\pm)}(p_r,\varphi)\right)^2}{Q_1^{(\pm)}(p_r,\varphi) - Q_2^{(\pm)}(p_r,\varphi)} \right), \quad (21)$$

with

$$Q_0^{(\pm)}(p_r,\varphi) = \tilde{q}_0^{(\pm)}(p_r,\varphi), \quad Q_1^{(\pm)}(p_r,\varphi) = \sum_{i=1}^{3} \tilde{q}_i^{(\pm)}(p_r,\varphi)\eta_i,$$
$$Q_2^{(\pm)}(p_r,\varphi) = \sum_{i,j=1,i\le j}^{3} \tilde{q}_{ij}^{(\pm)}(p_r,\varphi)\eta_i\eta_j. \quad (22)$$

This formula is valid for a general 3D tilted orthorhombic media. In this formula, the three anellipticity parameters control the accuracy of approximation. According to the definitions 2–4 for the three anellipticity parameters, the anellipticity parameters η_1 and η_2 control the accuracy of approximation in the two symmetry planes of a tilted orthorhombic medium, which correspond to the [y, z] and [x, z] planes for the orthorhombic media before rotation; the anellipticity parameter η_3 controls the accuracy of approximation between the two symmetry planes of the tilted orthorhombic media.

This formula has the following properties:

$$q^{(\pm)}(p_r, \varphi) = q^{(\pm)}(-p_r, \pi + \varphi), \quad (23)$$

$$q^{(\pm)}(p_r, \varphi) = -q^{(\mp)}(-p_r, \varphi). \quad (24)$$

In equation 23, (p_r, φ) and $(-p_r, \pi + \varphi)$ correspond to the same horizontal slowness components in Cartesian coordinates. Equation 24 implies that the upper and lower parts of slowness surface are antisymmetric with respect to the origin of slowness coordinate system. Equations 23 and 24 can be verified by considering equations 21 and 22, as well as the perturbation coefficients in Appendix A.

For 3D elliptically orthorhombic media ($\eta_1 = \eta_2 = \eta_3 = 0$), the first- and second-order coefficients vanish, that is, $Q_1^{(\pm)} = Q_2^{(\pm)} = 0$. Thus, approximation 21 becomes the exact formula

$$q^{(\pm)}(p_r, \varphi) = \frac{A(\varphi)\tilde{p}_r \pm \sqrt{\mathrm{v}^2(\varphi) - B(\varphi)\tilde{p}_r^2}}{v_{p0}\mathrm{v}^2(\varphi)}, \quad (25)$$

where $\tilde{p}_r = v_{p0}p_r$ denotes the radial horizontal slowness component normalized by the velocity v_{p0}; functions $A(\varphi)$, $B(\varphi)$, and $\mathrm{v}(\varphi)$ are given by equations A-2 to A-6 in Appendix A. This implies that the proposed formula 21 is accurate when the three anellipticity parameters are very small.

For a 2D TTI media, the anisotropy parameters defined in equations 2–6 become $r_1 = r_2$, $\eta_1 = \eta_2$, and $\eta_3 = 0$. By further setting $\phi = \varphi = 0$, the slowness rotation 13 becomes equation 2 in Stovas and Alkhalifah (2013). Furthermore, the proposed approximation is reduced to the one for 2D TTI media (Stovas and Alkhalifah, 2013).

DOWN- AND UPGOING PLANE WAVES FOR TILTED ELLIPTICALLY ORTHORHOMBIC MEDIA

For a tilted orthorhombic medium, the upper and lower parts of slowness surface do not exactly correspond to the downgoing and upgoing waves. Figure 2 shows an example of the slowness surface for a tilted elliptically orthorhombic medium. We can see that the upper and lower parts of the slowness surface intersect the axis $q = 0$. All plots in this figure are produced by using formula 25 for elliptically orthorhombic media with tilted symmetry axes. Let us now determine the range of horizontal slowness components of down- and upgoing plane P waves in tilted elliptically orthorhombic media. Figure 3 shows a schematic plot of the slowness surface composed of down- and upgoing plane waves with a fixed propagation azimuth. For a fixed propagation azimuth φ, the dipping direction of slowness surface is controlled by the sign of $A(\varphi)$. For tilted orthorhombic media, down- and upgoing plane waves consist of a part of the upper slowness surface and a part of the lower slowness surface. In Figure 3, points M and N correspond to the boundary points of horizontal slowness component for the upper and lower parts of the slowness surface. Points M and N are antisymmetric with respect to the origin of slowness coordinate system. Points P and Q are located on the horizontal slowness axis and correspond to the slowness of the horizontally propagating plane waves. Points P and Q are symmetric with respect to the vertical slowness axis. For a fixed propagation azimuth, it is not difficult to determine the slowness at points M, N, P, and Q from equation 25: for points M and N, we define $p_c(\varphi)$ and $q_c(\varphi)$ as the magnitudes of the radial horizontal and vertical slowness components

$$p_c(\varphi) \equiv \frac{\mathrm{v}(\varphi)}{\upsilon_{p0}\sqrt{B(\varphi)}}, \qquad q_c(\varphi) \equiv \frac{|A(\varphi)|}{\upsilon_{p0}\,\mathrm{v}(\varphi)\sqrt{B(\varphi)}}. \quad (26)$$

For points P and Q, we define $p_0(\varphi)$ as the magnitude of their slownesses

$$p_0(\varphi) \equiv \frac{\mathrm{v}(\varphi)}{\upsilon_{p0}\sqrt{A^2(\varphi) + B(\varphi)}}. \quad (27)$$

In equations 26 and 27, expressions for $A(\varphi)$, $B(\varphi)$, and $\mathrm{v}(\varphi)$ are given by equations A-2 to A-7 in Appendix A.

Using these definitions, the vertical slowness component of downgoing plane P waves, $q^{(D)}(p_r, \varphi)$, is determined by the following conditions:

If $A(\varphi) > 0$,

$$q^{(D)}(p_r,\varphi) = \begin{cases} q^{(+)}(p_r,\varphi)), & \text{for } p_r \in (-p_0(\varphi), p_c(\varphi)) \\ q^{(-)}(p_r,\varphi)), & \text{for } p_r \in (p_0(\varphi), p_c(\varphi)) \end{cases}, \quad (28)$$

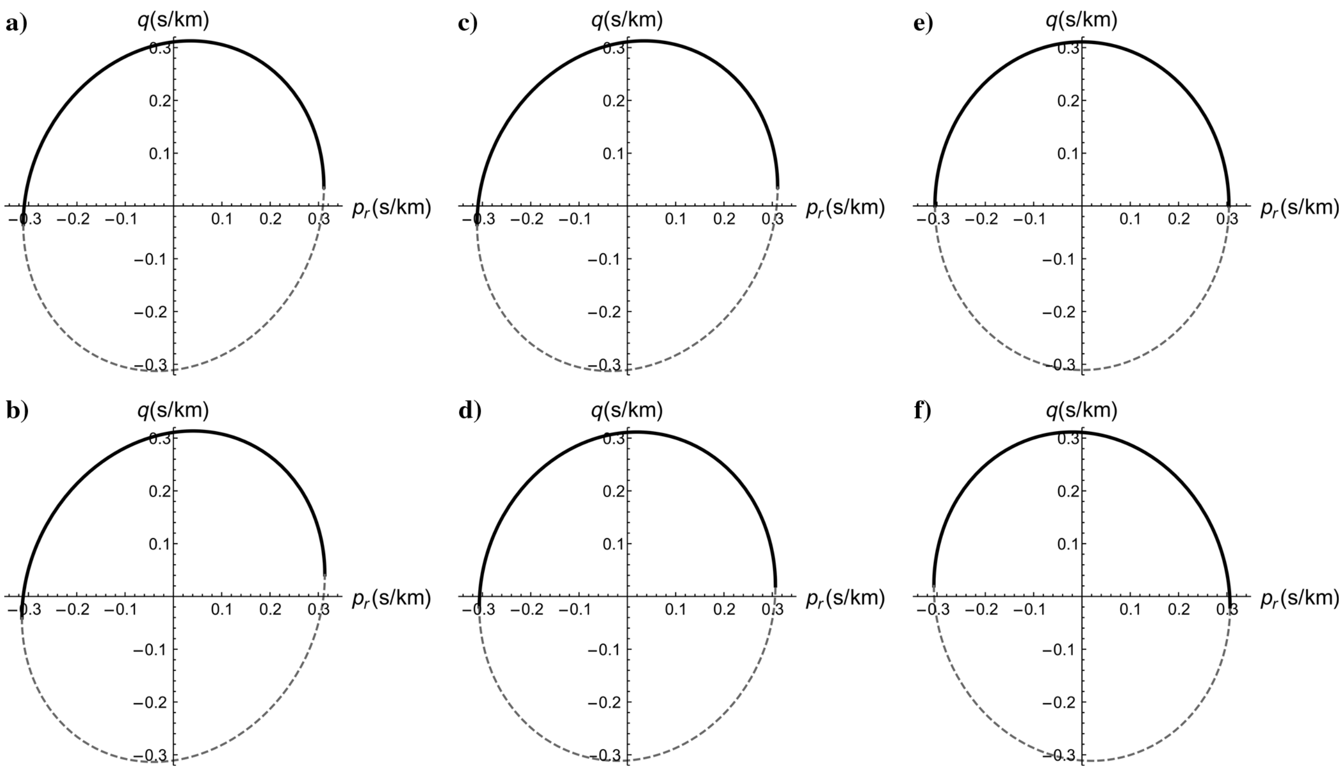

Figure 2. The slowness surface of P waves in an elliptically orthorhombic medium. Solid black lines are produced using equation 25 with plus sign in front of the square root. Dashed gray lines correspond to equation 25 with minus sign in front of the square root. From the top left to the bottom right, plots correspond to the azimuths of P-wave phase propagation direction (a) $\varphi = 0$, (b) $\varphi = \pi/6$, (c) $\varphi = \pi/3$, (d) $\varphi = \pi/2$, (e) $\varphi = 2\pi/3$, and (f) $\varphi = 5\pi/6$, respectively. The medium parameters include $\upsilon_{p0} = 3.0$ km/s, $r_1 = 1.2$ and $r_2 = 1.3$, $\eta_1 = \eta_2 = \eta_3 = 0$, $\theta = \pi/4$, $\phi = \pi/6$, and $\psi = 0$.

else if $A(\varphi) < 0$,

$$q^{(D)}(p_r, \varphi) = \begin{cases} q^{(+)}(p_r, \varphi)), & \text{for } p_r \in (-p_c(\varphi), p_0(\varphi)) \\ q^{(-)}(p_r, \varphi)), & \text{for } p_r \in (-p_c(\varphi), -p_0(\varphi)) \end{cases}, \tag{29}$$

else if $A(\varphi) = 0$,

$$q^{(D)}(p_r, \varphi) = q^{(+)}(p_r, \varphi), \quad \text{for } p_r \in (-p_0(\varphi), p_0(\varphi)). \tag{30}$$

Similarly, the vertical slowness component of upgoing plane waves, $q^U(p_r, \varphi)$ is determined by

If $A(\varphi) > 0$,

$$q^{(U)}(p_r, \varphi) = \begin{cases} q^{(-)}(p_r, \varphi)), & \text{for } p_r \in (-p_c(\varphi), p_0(\varphi)) \\ q^{(+)}(p_r, \varphi)), & \text{for } p_r \in (-p_c(\varphi), -p_0(\varphi)) \end{cases}, \tag{31}$$

else if $A(\varphi) < 0$,

$$q^{(U)}(p_r, \varphi) = \begin{cases} q^{(-)}(p_r, \varphi)), & \text{for } p_r \in (-p_0(\varphi), p_c(\varphi)) \\ q^{(+)}(p_r, \varphi)), & \text{for } p_r \in (p_0(\varphi), p_c(\varphi)) \end{cases}, \tag{32}$$

else if $A(\varphi) = 0$,

$$q^{(U)}(p_r, \varphi) = q^{(-)}(p_r, \varphi), \quad \text{for } p_r \in (-p_0(\varphi), p_0(\varphi)). \tag{33}$$

The conditions presented above are strictly exact for elliptically orthorhombic media. Because the proposed formula in the previous section is based on the perturbation of media from a tilted elliptically orthorhombic background, we can use the conditions to calculate the vertical slowness components of down- and upgoing plane waves for general tilted orthorhombic media. However, the examples in section "Numerical results" will show that the conditions are not strictly suitable for general tilted orthorhombic media. For instance, Figure 4 shows the nearly vertical dashed lines, which correspond to the extremely large error in vertical slowness. We call it the "jumping" phenomenon throughout this paper for simplicity.

VALIDITY REGION FOR TILTED ORTHORHOMBIC MEDIA

To overcome the jumping phenomenon mentioned above, we present a validity range of the radial horizontal slowness for formula 21. The multivalued range of vertical slowness component for tilted elliptically orthorhombic media is excluded from the validity range because the "Numerical results" section will show the multivalued region causes the significantly error in the vertical slowness component for formula 21.

For a fixed propagation azimuth φ, the two intersections of the exact slowness surface with the horizontal plane $q = 0$, are defined as the boundary of the validity range of the horizontal slowness components of down- and upgoing plane waves. Finally, the vertical slowness components of down- and upgoing plane waves are given by as follows,

for downgoing plane waves,

$$q^{(D)}(p_r, \varphi) = q^{(+)}(p_r, \varphi)), \quad \text{for } p_r \in (-p_e(\varphi), p_e(\varphi)), \tag{34}$$

for upgoing plane waves,

$$q^{(U)}(p_r, \varphi) = q^{(-)}(p_r, \varphi)), \quad \text{for } p_r \in (-p_e(\varphi), p_e(\varphi)), \tag{35}$$

where $p_e(\varphi)$ denotes the intersection of the exact slowness surface with the horizontal slowness plane $q = 0$, identical to the magnitude of the slowness of the horizontally propagating plane P waves, and its expression is derived in Appendix B; $q^{(+)}$ and $q^{(-)}$ denote the upper and lower parts of the slowness surface, given in equation 21 with 22.

Although not strictly proven, our experience shows the validity range given in equations 34 and 35 is fully located inside of the range of the radial horizontal slowness components for general tilted orthorhombic media. We can also express this relation by the following equation:

$$(-p_e(\varphi), p_e(\varphi)) \subset (-p_c(\varphi), p_c(\varphi)), \tag{36}$$

where the meaning of $p_e(\varphi)$ is explained after equation 35; the definition of $p_c(\varphi)$ is given by the first of equation 26.

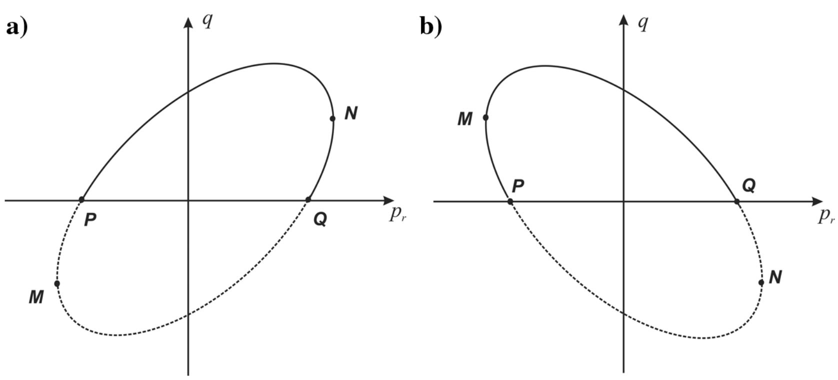

Figure 3. The schematic plots of slowness surface composed of downgoing (solid line) and upgoing (dashed line) plane P waves with propagation azimuth φ in tilted elliptically orthorhombic media. The left and right plots correspond to the cases $A(\varphi) > 0$ and $A(\varphi) < 0$, respectively. Points M and N are the boundary points of the upper (or lower) part of the slowness surface for a fixed φ, which means that the tangents at points M and N are parallel to the vertical slowness axis q. Points P and Q are the intersections of the slowness surface with $q = 0$ for a fixed φ.

NUMERICAL RESULTS

In this section, we study the validity of the proposed formula by comparing with the exact solution that is obtained by mapping the corresponding phase velocity on the slowness surface. The exact phase-velocity of plane P waves is shown in Tsvankin (1997) for elastic orthorhombic media. For acoustic orthorhombic media in this paper, the phase velocity can also be calculated in his procedure by considering the relationships 1–6 between stiffness coefficients and the parameters in the modified Alkhalifah's (2003) notation. To check the influence of anellipticity on the accuracy of the proposed formula, we use two tilted orthorhombic models with same parameters $v_{p0} = 3.0$ km/s, $r_1 = 1.2$ and $r_2 = 1.3$, $\theta = \pi/4$, $\phi = \pi/6$, and $\psi = 0$, but different anellipticities η_i, $i = 1, 2, 3$. The anellipticity parameters for the first model are $\eta_1 = 0.2$, $\eta_2 = 0.1$, and $\eta_3 = 0.3$; for the second model $\eta_1 = \eta_2 = \eta_3 = 0.3$.

Let us now analyze the error of formula 21. The conditions in equations 28–33 are accounted for the range of the radial horizontal slowness of down- and upgoing plane waves. For the first model with relatively weak anellipticity, Figures 4 and 5 show the vertical slowness curves of down- and upgoing plane P waves, respectively. We can see that the results from formula 21 match very well with the exact ones when the propagation direction is far away from the horizontal direction. The jumping phenomenon of the vertical slowness curves happens due to two reasons: One is when the propagation direction of plane P waves is close to the point of tangency (p_c, q_c); the other possible reason is that when the denominator in equation 21 is so small that the error of Shanks transformation is amplified. Figures 6 and 7 show results for the second orthorhombic model with relatively strong anellipticity. Similar to for the previous model, the jumping phenomenon still exists. Compared with Figures 4 and 5, Figures 6 and 7 show the validity range of propagation angle becomes narrowed for the orthorhombic medium with relatively strong anellipticity. The decrease in validity range of the propagation angles is due to the increase in anellipticity, because the proposed formula is derived from the vertical slowness expansion in terms of the three anellipticity parameters. This proves that the three anellipticity parameters control the accuracy of the proposed formula.

To exclude the jumping phenomenon mentioned above, let us now consider the validity range of the radial horizontal slowness components given in equations 34 and 35 for downgoing and upgoing plane waves when using the proposed formula. To measure the validity range of the propagation angles, we define the polar angle ϑ as an acute angle of the phase-propagation direction of plane P waves and the z-axis of the tilted orthorhombic medium. Figures 8 and 9 correspond to Figures 4 and 5, respectively, for the first orthorhombic model with relatively weak anellipticity. The only difference is that for Figures 8 and 9, we consider the validity range given in equations 34 and 35 to calculate the vertical slowness components of downgoing and upgoing plane waves. We can see that the validity range of the propagation angles ϑ is azimuthal dependent. Considering all azimuths shown in Figures 4 and 5, the average validity range of polar angle is approximately $\vartheta \in (-70°, 70°)$. In the validity range, the proposed formula is very close to the exact solution; the jumping phenomenon disappears because the slowness outside the validity range is not considered. Similar results are shown in Figures 10 and 11 for the second orthorhombic model with relatively strong anellipticity. For all propagation azimuths, we coarsely measure the average validity range of polar angles, $\vartheta \in (-67^{\circ}, 67^{\circ})$, within which formula 21 is valid for all propagation azimuth. When the polar angle ϑ is outside the validity region, $\vartheta \notin (-67^{\circ}, 67^{\circ})$, the proposed formula is less accurate. Comparing the validity ranges of polar angles for the first and second orthorhombic models, we can find that the magnitude of anellipticity affects the validity ranges of polar angles: The stronger the anellipticity of the orthorhombic medium, the smaller the validity range of polar angles. However, it is difficult to describe how the three anellipticity parameters affect the range of polar angles, because the three Euler angles for a tilted orthorhombic medium are also included in our approximation for the vertical slowness component. Comparing Figures 10 and 11 with Figures 8 and 9, we can see that the influence of anellipticity on the accuracy of the proposed approximation is nor pronounced in the validity range. Therefore, the proposed formula applies for tilted orthorhombic media

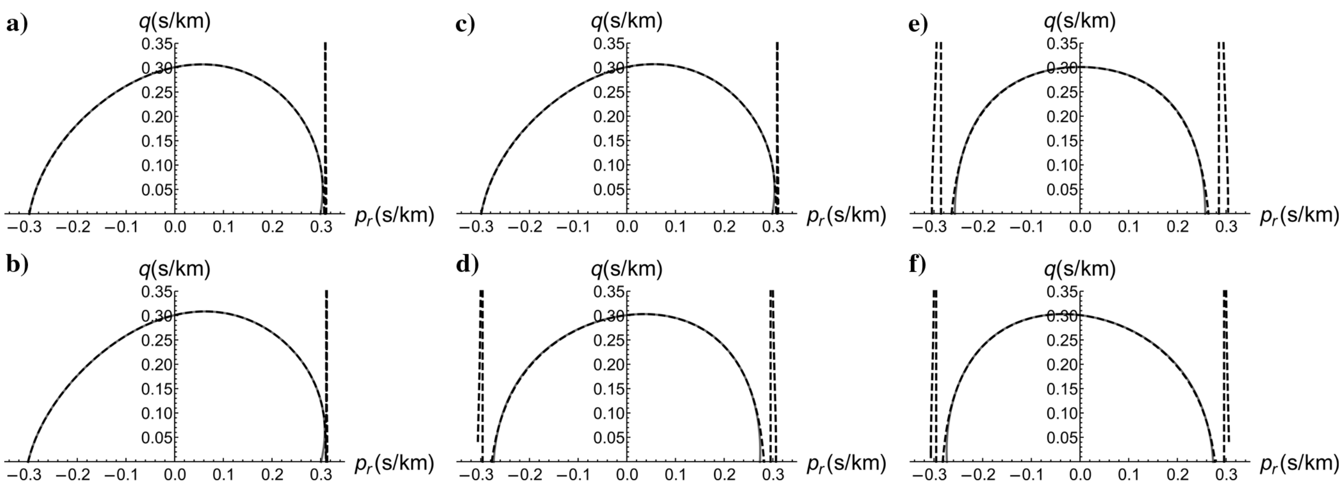

Figure 4. The vertical slowness component of downgoing plane P waves versus the horizontal slowness along different azimuths of phase propagation direction in an orthorhombic medium. Solid gray lines correspond to the exact solution, and dashed black lines correspond to the proposed formula. From the top left to the bottom right, plots correspond to the azimuths of plane P-wave phase propagation direction (a) $\varphi = 0$, (b) $\varphi = \pi/6$, (c) $\varphi = \pi/3$, (d) $\varphi = \pi/2$, (e) $\varphi = 2\pi/3$, and (f) $\varphi = 5\pi/6$, respectively. The medium parameters include $v_{p0} = 3.0$ km/s, $r_1 = 1.2$ and $r_2 = 1.3$, $\eta_1 = 0.2$, $\eta_2 = 0.1$, $\eta_3 = 0.3$, $\theta = \pi/4$, $\phi = \pi/6$, and $\psi = 0$.

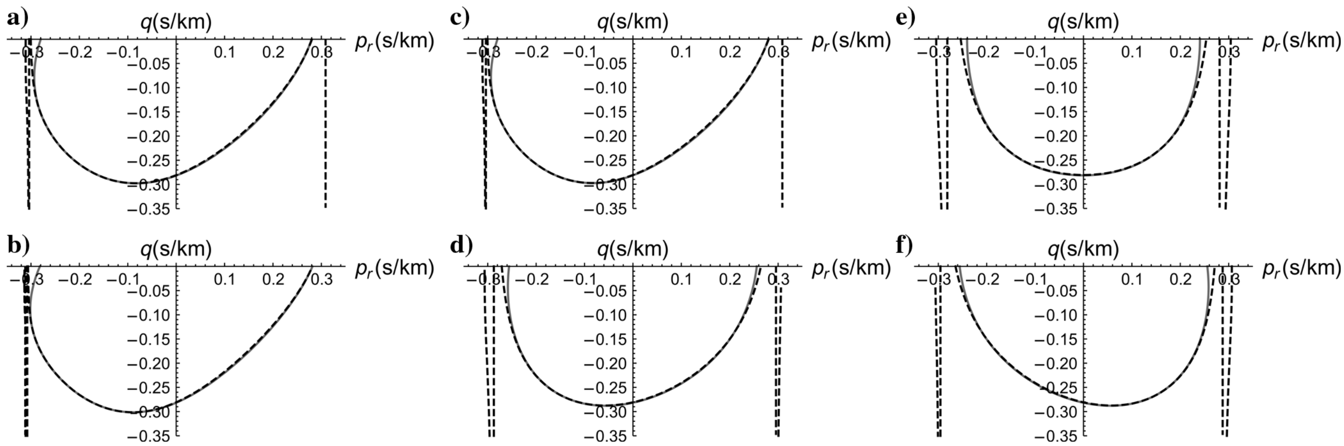

Figure 5. Similar to Figure 4, but for upgoing P waves.

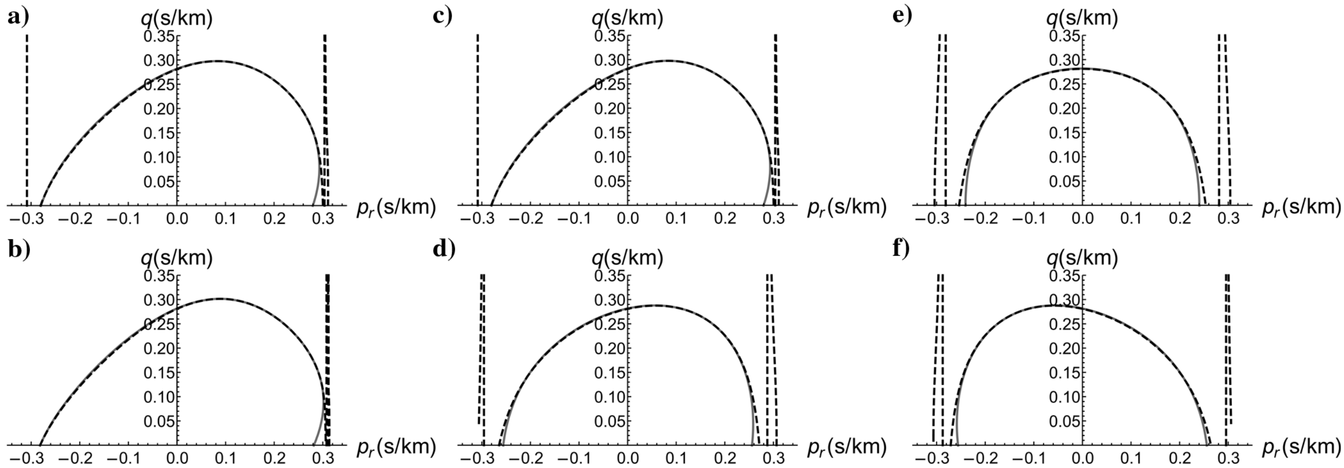

Figure 6. Similar to Figure 4, but the anellipticity parameters are $\eta_1 = 0.3$, $\eta_2 = 0.3$, and $\eta_3 = 0.3$.

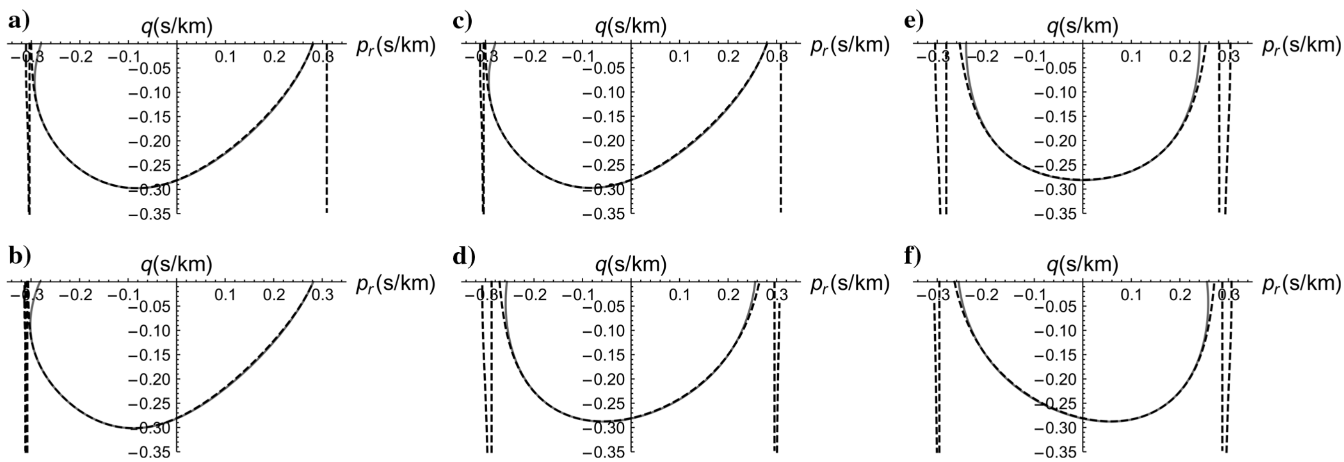

Figure 7. Similar to Figure 5, but the medium parameters are $v_{p0} = 3.0$ km/s, $r_1 = 1.2$ and $r_2 = 1.3$, $\eta_1 = 0.3$, $\eta_2 = 0.3$, $\eta_3 = 0.3$, $\theta = \pi/4$, $\phi = \pi/6$, and $\psi = 0$.

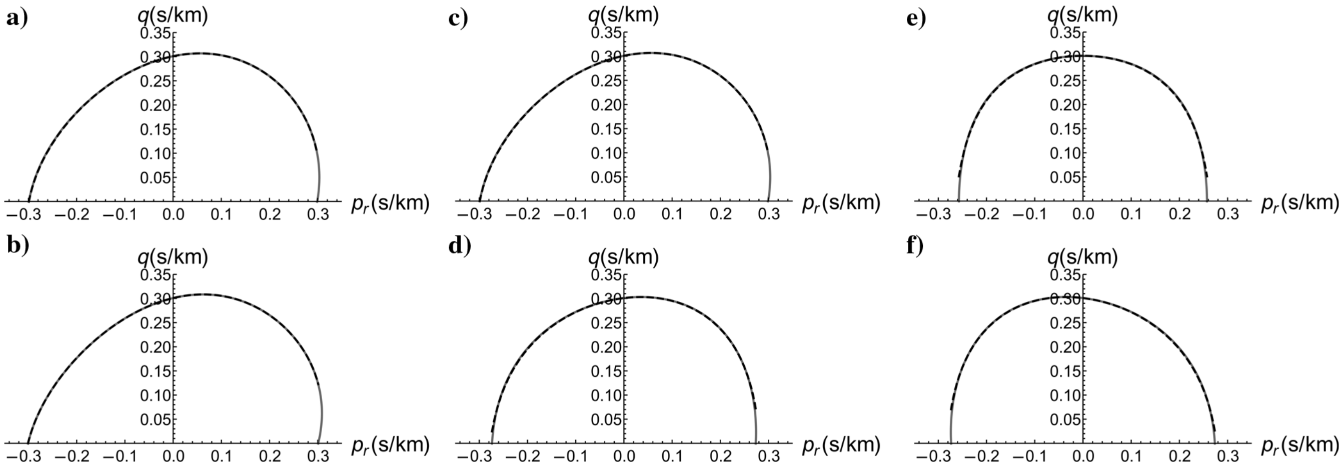

Figure 8. Same as Figure 4, except the validity range of the radial horizontal slowness components is considered.

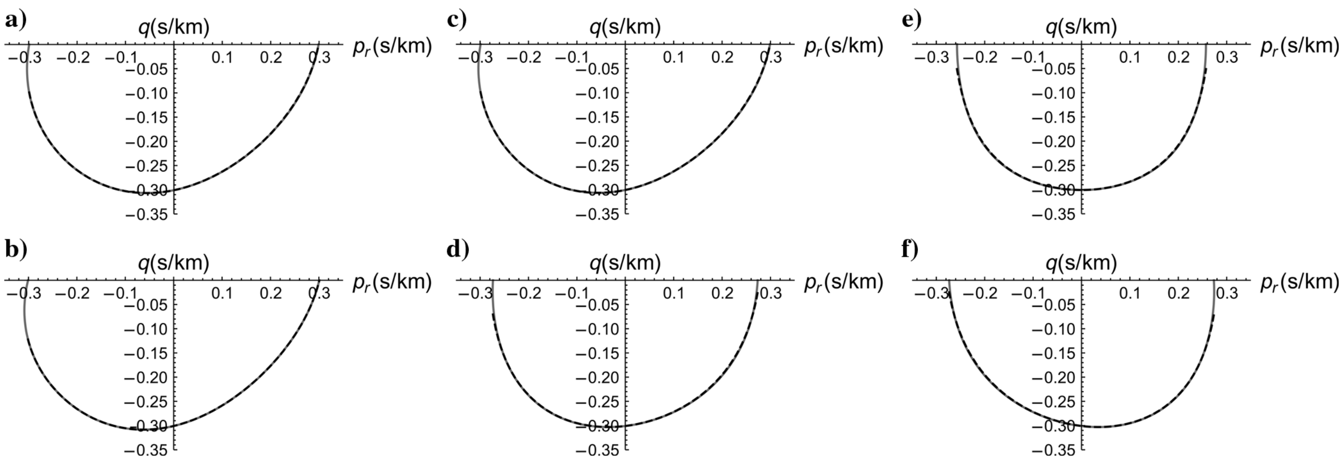

Figure 9. Same as Figure 5, except the validity range of the radial horizontal slowness components is considered.

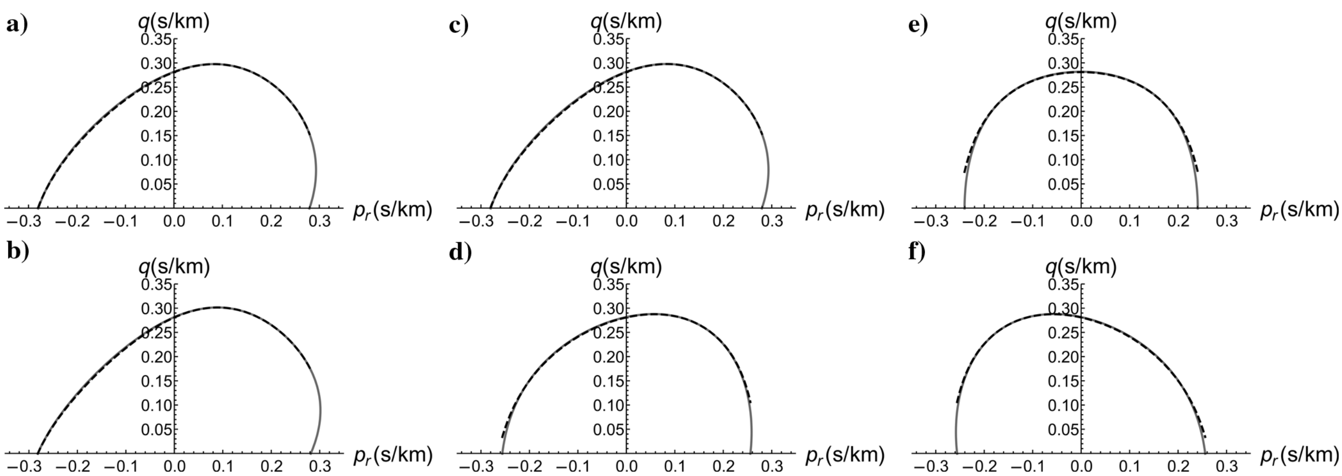

Figure 10. Same as Figure 6, except the validity range of the radial horizontal slowness components is considered.

with strong anellipticity, provided that horizontal slowness component is located in the validity range given by equations 34 and 35.

DISCUSSION

Because the proposed formula is analytic, it is very easily implemented in complex procedures, such as seismic ray tracing and phase-shift migration. The disadvantage of the proposed formula is that it cannot achieve a wide-angle approximation for tilted orthorhombic media. The "Numerical results" secton illustrates that the validity range of polar angles is approximately $\vartheta \in (-67^{\circ}, 67^{\circ})$ for the second orthorhombic model with relatively strong anellipticity. Therefore, the validity range of the radial horizontal slowness component must be considered before using the proposed formula.

In the "Introduction," we mention that it is useful for applied seismology to represent the vertical slowness component in terms of the horizontal components for anisotropic media. We now discuss some specific applications of the proposed formula. First, the proposed formula can be used in phase-shift migration of surface P-wave data for 3D tilted orthorhombic media. From the relationship between slowness and wavenumber, our formula for the vertical slowness component can be easily transformed to calculate the vertical wavenumber component. For laterally homogeneous media, the phase-shift extrapolation of source and receiver wavefield is discussed in Berkhout (1982, pp. 225–240). For lateral heterogeneous media, Gazdag and Squazzero (1984) propose the phase-shift migration plus interpolation. Margrave and Ferguson (1999) propose the pseudodifferential equation for phase-shift migration. It is not difficult to extend their procedures for 3D tilted orthorhombic media with lateral variations. Second, the proposed formula can be used as Snell's law to calculate the vertical slowness component of reflected and transmitted P waves in layered orthorhombic media. In the case of a horizontally layered orthorhombic medium, the lateral slowness is preserved for all layers, and the proposed formula yields the vertical slowness for the down- and upgoing waves for each layer. Snell's law requires the lateral projections of slownesses of all generated waves on the tangent of a local interface are equal to that of the incident wave, which must be taken in account for ray tracing in multilayered media. Numerical examples show that our formula is very accurate for incident angles up to 67° in tilted orthorhombic media with strong anellipticity. This means that our formula can satisfy the requirement of calculating reflection traveltimes of P waves with short to moderate source-receiver offsets in multilayered orthorhombic media. Last, let us discuss in detail the application of the proposed formula on calculating the intercept time in τ-p domain for horizontally layered, tilted orthorhombic media. In this case, for a fixed azimuth of phase propagation direction, the radial horizontal slowness is preserved for all horizontal layers. The intercept time in τ-p domain is expressed by the summation over the product of layer thickness and absolute magnitudes of vertical slowness components of down- and upgoing waves

$$\tau(p_r, \varphi) = \sum_{i=1}^{N} z_i (q^{(D)}(p_r, \varphi) - q^{(U)}(p_r, \varphi)), \qquad (37)$$

where τ denotes the intercept time; p_r denotes the radial horizontal slowness; φ denotes the azimuth of radial horizontal slowness; $q^{(D)}$

Table 1. Parameters for a five-layer orthorhombic model: Δz denotes the layer thickness; v_{p0} denotes the P-wave velocity along the z-axis in orthorhombic media with the three symmetry planes orthogonal to the Cartesian coordinates; r_1 and r_2 are the factors for the NMO velocities squared defined in the [y, z] and [x, z] planes in the orthorhombic media; η_1, η_2, and η_3 are three anellipticity parameters defined in the [y, z], [x, z], and [x, y] planes of the orthorhombic media; and ϕ, θ, and ψ are Euler angles introduced in Figure 1 to describe the orientation of a tilted orthorhombic medium with respect to an orthorhombic media with the three symmetry planes orthogonal to the Cartesian coordinates.

Layer	Δz (km)	v_{p0} (km/s)	r_1	r_2	η_1	η_2	η_3	ϕ	θ	ψ
1	0.3	2.0	1.1	1.2	0.1	0.2	0.1	0	0	0
2	0.4	2.5	1.2	1.1	0.05	0.1	0.1	$\pi/6$	0	$\pi/3$
3	0.5	3.0	1.0	1.2	0.1	0.2	0.05	$\pi/4$	$\pi/6$	$\pi/4$
4	0.45	3.5	1.3	1.2	0.05	0.2	0.15	$\pi/3$	$\pi/4$	$\pi/6$
5	0.55	4.0	1.3	1.15	0.05	0.05	0.25	$\pi/6$	$\pi/3$	$\pi/3$

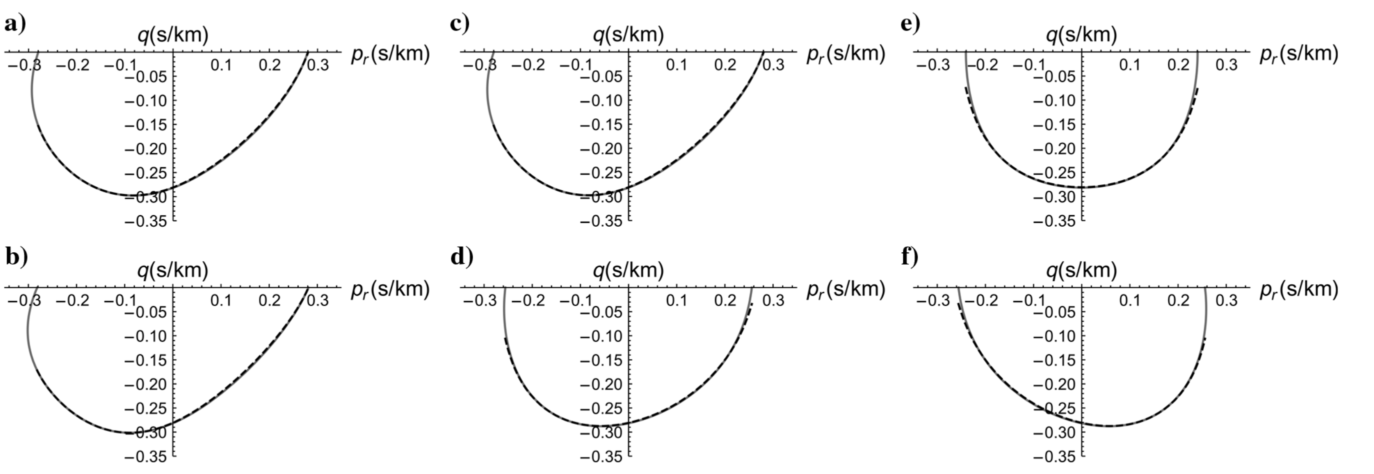

Figure 11. Same as Figure 7, except the validity range of the radial horizontal slowness components is considered.

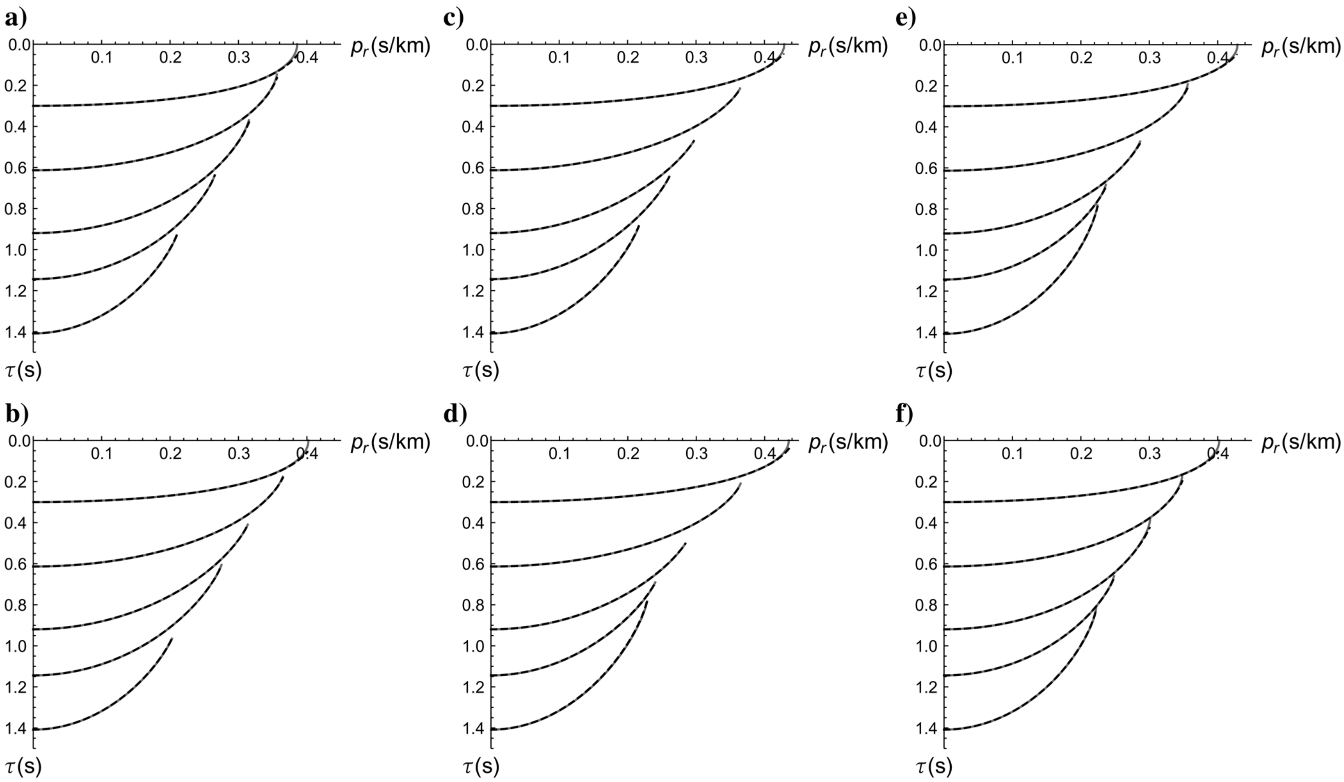

Figure 12. The P-wave intercept time in τ-p domain versus the radial horizontal slowness component along different azimuths of the phase propagation direction in a five-layer orthorhombic model. Dashed black lines correspond to the approximate intercept times calculated by the proposed vertical slowness formula, and solid gray lines correspond to the exact intercept times. From the top left to the bottom right, the plots correspond to the azimuths of phase propagation direction (a) $\varphi = 0$, (b) $\varphi = \pi/6$, (c) $\varphi = \pi/3$, (d) $\varphi = \pi/2$, (e) $\varphi = 2\pi/3$, and (f) $\varphi = 5\pi/6$, respectively.

and $q^{(U)}$ denote the vertical slowness components of down- and upgoing P waves, which can be approximately calculated by equations 34 and 35; and z_i denotes the thickness of the ith layer.

From equations 23 and 24, we find that the intercept time satisfies the following symmetry properties:

$$\tau(p_r, \varphi) = \tau(p_r, \pi + \varphi). \tag{38}$$

Combining equations 37 and 21, we calculate the approximate intercept time for a five-layer orthorhombic model. The interval parameters of this model are shown in Table 1. The validity regions 34 and 35 are considered when using formula 21 to calculate the approximate vertical slowness components of down- and upgoing P waves. Figure 12 shows the approximate intercept times with the exact ones. This example indicates that the proposed formula for vertical slowness is applicable for calculating intercept time in τ-p domain for horizontally layered, tilted orthorhombic media.

CONCLUSION

A perturbation method and Shanks transform are combined to derive the proposed formula for the vertical slowness components of down- and upgoing plane P waves in tilted orthorhombic media. The validity range for the proposed formula is determined by the slowness of the horizontally propagating plane P waves in tilted orthorhombic media. The proposed formula is accurate for tilted orthorhombic media with weak to strong anellipticity.

ACKNOWLEDGMENT

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APPENDIX A

EXPANSION COEFFICIENTS OF VERTICAL SLOWNESS COMPONENT

In this appendix, we show the perturbation coefficients defined in equation 20. The zero-order coefficient $\tilde{q}_0$ corresponding to an elliptically orthorhombic medium $(\eta_i = 0, i = 1,2,3)$ with tilted symmetry axes is given by

$$\tilde{q}_0^{(\pm)}(p_r, \varphi) = \frac{A(\varphi)\tilde{p}_r \pm \sqrt{\mathrm{v}^2(\varphi) - B(\varphi)\tilde{p}_r^2}}{\mathrm{v}^2(\varphi)}, \tag{A-1}$$

where $\tilde{p}_r = v_{p0} p_r$; functions $A(\varphi)$, $B(\varphi)$, and $\mathrm{v}(\varphi)$ are given by

$$A(\varphi) = \frac{1}{2}(r_1 - r_2)\sin(2\psi)\sin\theta\sin(\phi - \varphi) - \frac{1}{4}(2 - r_1 - r_2 + (r_1 - r_2)\cos(2\psi))\sin(2\theta)\cos(\phi - \varphi), \tag{A-2}$$

$$\mathrm{v}(\varphi) = \cos^2\theta + r_2\cos^2\psi\sin^2\theta + r_1\sin^2\theta\sin^2\psi, \quad \text{(A-3)}$$

$$B(\varphi) = B_0 + B_1\cos(2(\phi-\varphi)) + B_2\sin(2(\phi-\varphi)), \quad \text{(A-4)}$$

with

$$B_0 = \frac{1}{8}(3r_1 + 3r_2 + 2r_1r_2) + \frac{1}{8}(r_1 + r_2 - 2r_1r_2)\cos^2\theta - \frac{1}{8}(r_1 + r_2 - 2r_1r_2 + 2(r_1 - r_2)\cos(2\psi))\sin^2\theta, \quad \text{(A-5)}$$

$$B_1 = -\frac{1}{8}(r_1 - r_2)(3 + \cos(2\theta))\cos(2\psi) + \frac{1}{4}(r_1 + r_2 - 2r_1r_2)\sin^2\theta, \quad \text{(A-6)}$$

$$B_2 = (r_1 - r_2)\cos\theta\cos\psi\sin\psi. \quad \text{(A-7)}$$

By considering the following definitions related to the slowness components for an orthorhombic medium with the three symmetry planes orthogonal to the Cartesian coordinates

$$\tilde{p}_{\mathrm{v}1} \equiv \tilde{p}_r(\cos\theta\cos(\phi-\varphi)\cos\psi - \sin(\phi-\varphi)\sin\psi) - \tilde{q}_0\cos\psi\sin\theta, \quad \text{(A-8)}$$

$$\tilde{p}_{\mathrm{v}2} \equiv -\tilde{p}_r(\cos\psi\sin(\phi-\varphi) + \sin\psi\cos\theta\cos(\phi-\varphi)) + \tilde{q}_0\sin\theta\sin\psi, \quad \text{(A-9)}$$

$$\tilde{q}_{\mathrm{v}} \equiv \tilde{p}_r\cos(\phi-\varphi)\sin\theta + \tilde{q}_0\cos\theta, \quad \text{(A-10)}$$

where equations A-8 to A-10 follows from equation 13 with $\boldsymbol{p_v} = (\tilde{p}_{\mathrm{v}1}, \tilde{p}_{\mathrm{v}2}, \tilde{q}_{\mathrm{v}})^T$ and $\boldsymbol{p} = (\tilde{p}_r\cos\varphi, \tilde{p}_r\sin\varphi, \tilde{q}_0)^T$; $\tilde{q}_0$ is given by equation A-1, the first- and second-order coefficients $\tilde{q}_i$ and $\tilde{q}_{ij}$ in equation 20 are expressed in compact forms as follows:

The first-order coefficients $\tilde{q}_i$, $i = 1,2,3$

$$\tilde{q}_1^{(\pm)}(p_r,\varphi) = \mp\frac{r_1\tilde{p}_{\mathrm{v}2}^2(1-\tilde{q}_{\mathrm{v}}^2)}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-11)}$$

$$\tilde{q}_2^{(\pm)}(p_r,\varphi) = \mp\frac{r_2\tilde{p}_{\mathrm{v}1}^2(1-\tilde{q}_{\mathrm{v}}^2)}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-12)}$$

$$\tilde{q}_3^{(\pm)}(p_r,\varphi) = \mp\frac{\tilde{q}_{\mathrm{v}}^2 - (1 - r_1\tilde{p}_{\mathrm{v}2}^2)(1 - r_2\tilde{p}_{\mathrm{v}1}^2)}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}. \quad \text{(A-13)}$$

The second-order coefficients $\tilde{q}_{ij}$, $i, j = 1,2,3$, and $i \le j$

$$\tilde{q}_{11}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_1^2 - c_{11}\tilde{q}_1 - d}{2\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-14)}$$

$$\tilde{q}_{22}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_2^2 - c_{22}\tilde{q}_2 - d}{2\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-15)}$$

$$\tilde{q}_{33}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_3^2 - c_{33}\tilde{q}_3 - d}{2\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-16)}$$

$$\tilde{q}_{12}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_1\tilde{q}_2 - c_{12}\tilde{q}_1 - c_{21}\tilde{q}_2 + d}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-17)}$$

$$\tilde{q}_{13}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_1\tilde{q}_3 - c_{13}\tilde{q}_1 - c_{31}\tilde{q}_3 + e_{13}}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-18)}$$

$$\tilde{q}_{23}^{(\pm)}(p_r,\varphi) = \mp\frac{\mathrm{v}^2\tilde{q}_2\tilde{q}_3 - c_{23}\tilde{q}_2 - c_{32}\tilde{q}_3 + e_{23}}{\sqrt{\mathrm{v}^2 - B\tilde{p}_r^2}}, \quad \text{(A-19)}$$

with

$$c_{11} = 4\tilde{p}_{\mathrm{v}2}r_1(\tilde{p}_{\mathrm{v}2}\tilde{q}_{\mathrm{v}}\cos\theta - (1-\tilde{q}_{\mathrm{v}}^2)\sin\theta\sin\psi), \quad \text{(A-20)}$$

$$c_{22} = 4\tilde{p}_{\mathrm{v}1}r_2(\tilde{p}_{\mathrm{v}1}\tilde{q}_{\mathrm{v}}\cos\theta + (1-\tilde{q}_{\mathrm{v}}^2)\sin\theta\cos\psi), \quad \text{(A-21)}$$

$$c_{33} = -4(\tilde{q}_{\mathrm{v}}\cos\theta + \sin\theta(\tilde{p}_{\mathrm{v}2}r_1(1 - \tilde{p}_{\mathrm{v}1}^2r_2)\sin\psi - \tilde{p}_{\mathrm{v}1}r_2(1 - \tilde{p}_{\mathrm{v}2}^2r_1)\cos\psi)), \quad \text{(A-22)}$$

$$c_{12} = c_{32} = \frac{c_{22}}{2}, \quad \text{(A-23)}$$

$$c_{21} = c_{31} = \frac{c_{11}}{2}, \quad \text{(A-24)}$$

$$c_{13} = c_{23} = \frac{c_{33}}{2}, \quad \text{(A-25)}$$

$$d = r_1r_2\tilde{p}_{\mathrm{v}1}^2\tilde{p}_{\mathrm{v}2}^2\tilde{q}_{\mathrm{v}}^2, \quad \text{(A-26)}$$

$$e_{13} = r_1 \tilde{p}_{v2}^2 (2 - 2r_2 \tilde{p}_{v1}^2 - \tilde{q}_v^2 (2 - r_2 \tilde{p}_{v1}^2)), \tag{A-27}$$

$$e_{23} = r_2 \tilde{p}_{v1}^2 (2 - 2r_1 \tilde{p}_{v2}^2 - \tilde{q}_v^2 (2 - r_1 \tilde{p}_{v2}^2)). \tag{A-28}$$

Note that for convenience, we omit the arguments p_r and φ of functions $\tilde{p}_{v1}$, $\tilde{p}_{v2}$, $\tilde{q}_v$, and $\tilde{q}_i$ ($i = 1,2,3$) in the right sides of equations A-11 to A-28.

APPENDIX B

THE SLOWNESS OF HORIZONTALLY PROPAGATING PLANE P WAVES IN TILTED ORTHORHOMBIC MEDIA

In this appendix, we derive the exact slowness of horizontally propagating plane P waves in a tilted orthorhombic medium. For a fixed propagation azimuth φ, the vector of horizontal propagation direction in this medium is given by

$$\boldsymbol{n} = (\cos\,\varphi, \sin\,\varphi, 0)^T. \tag{B-1}$$

Considering the coordinate rotation 13, we find the vector of propagation direction, $n_v = (n_{v1}, n_{v2}, n_{v3})^T$, defined in the corresponding vertical orthorhombic medium

$$\boldsymbol{n}_v = \boldsymbol{R}_c \boldsymbol{R}_b \boldsymbol{R}_a \boldsymbol{n}, \tag{B-2}$$

where $\boldsymbol{R}_a$, $\boldsymbol{R}_b$, and $\boldsymbol{R}_c$ are given by equation 12.

The slowness components of plane P-wave in direction $\boldsymbol{n}_v$ is expressed by the magnitude of phase velocity υ

$$\boldsymbol{p}_v = \frac{1}{\upsilon} \boldsymbol{n}_v. \tag{B-3}$$

Substituting relation B-3 into the slowness surface equation 7 with equations 8–11 leads to the following cubic equation in υ^2:

$$\upsilon^6 + k_4 \upsilon^4 + k_2 \upsilon^2 + k_0 = 0, \tag{B-4}$$

with

$$\begin{aligned} k_4 = \upsilon_{p0}^2 (&-r_2(1+2\eta_2) + n_{v3}^2(-1 + r_2(1+2\eta_2)) \\ &- n_{v2}^2 (r_1(1+2\eta_1) + r_2(1+2\eta_2))), \end{aligned} \tag{B-5}$$

$$\begin{aligned} k_2 = \frac{2}{1+2\eta_3} \upsilon_{p0}^4 (&n_{v3}^2 (n_{v1}^2 r_2 \eta_2 + n_{v2}^2 r_1 \eta_1)(1+2\eta_3) \\ &+ n_{v1}^2 n_{v2}^2 r_1 r_2 (1+2\eta_1)(1+2\eta_2)\eta_3), \end{aligned} \tag{B-6}$$

$$k_0 = 2 n_1^2 n_2^2 n_3^2 r_1 r_2 \upsilon_{p0}^6 \Omega, \tag{B-7}$$

where Ω is given by equation 10 with 11.

Solving this cubic equation gives the slowness $p_e = 1/\upsilon$ of horizontally propagating plane P-wave along the propagation azimuth φ in a tilted orthorhombic medium

$$\frac{1}{p_e^2} = 2\mu \sqrt{\frac{1}{3}\left(\frac{k_4^2}{3} - k_2\right)} - \frac{k_4}{3}, \tag{B-8}$$

where

$$\mu = \cos\left(\frac{1}{3} \arccos\left(-\frac{\mu_1}{\mu_2}\right)\right), \tag{B-9}$$

with

$$\mu_1 = 2\left(\frac{k_4}{3}\right)^3 - \frac{k_4 k_2}{3} + k_0, \tag{B-10}$$

$$\mu_2 = 2\left(\frac{1}{3}\left(\frac{k_4^2}{3} - k_2\right)\right)^{3/2}. \tag{B-11}$$

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